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Geometric description of clustering in directed networks

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CONTENTS

S.I. CONTROLLING THE RECIPROCITY IN RANDOM DIRECTED NETWORKS

We consider a general random directed networks model in which p_{ij} is the probability for a directed link to exist from node i to node j. We denote the number of nodes with N. To control the level of reciprocity, our approach focuses on each *pair* of directed links between two nodes and defines the four following symmetrical probabilities

$$
\begin{aligned}
&P_{ij}(a_{ij}=0,a_{ji}=0) && \text{(none of the two possible directed links exist)}\\
&P_{ij}(a_{ij}=1,a_{ji}=0) && \text{(the link from node } i \text{ to node } j \text{ exists but the other does not)}\\
&P_{ij}(a_{ij}=0,a_{ji}=1) && \text{(the link from node } j \text{ to node } i \text{ exists but the other does not)}\\
&P_{ij}(a_{ij}=1,a_{ji}=1) && \text{(both directed links exists)}
\end{aligned}
$$

with $1 \leq i < j \leq N$ and where a_{ij} is 1 if there is a directed link from node i to node j, and 0 otherwise. These four joint probabilities are normalized

$$P_{ij}(a_{ij}=0,a_{ji}=0) + P_{ij}(a_{ij}=1,a_{ji}=0) + P_{ij}(a_{ij}=0,a_{ji}=1) + P_{ij}(a_{ij}=1,a_{ji}=1) = 1 \ , \tag{S1}$$

and their marginals must be coherent with the random directed network model

$$P_{ij}(a_{ij}=1,a_{ji}=0) + P_{ij}(a_{ij}=1,a_{ji}=1) = p_{ij} \ , \tag{S2a}$$

$$P_{ij}(a_{ij}=0,a_{ji}=1) + P_{ij}(a_{ij}=1,a_{ji}=1) = p_{ij} \ . \tag{S2b}$$

To connect the probabiilities $P_{ij}(a_{ij},a_{ji})$ with the reciprocity in the network ensemble defined by the model, we look at the following correlation coefficient

$$\rho_{ij} = \frac{\langle a_{ij}a_{ji}\rangle - \langle a_{ij}\rangle\langle a_{ji}\rangle}{\sqrt{\left(\langle a_{ij}^2\rangle - \langle a_{ij}\rangle^2\right)\left(\langle a_{ji}^2\rangle - \langle a_{ji}\rangle^2\right)}} = \frac{P_{ij}(1,1) - p_{ij}p_{ji}}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \ . \tag{S3}$$

for each pair (i,j) with $1 \leq i < j \leq N$, and where $\langle\cdot\rangle$ corresponds to an average over the network ensemble. A closed form for $P_{ij}(1,1)$ in terms of p_{ij} and p_{ji} can be obtained by combining Eqs. (S2)–(S3) alongside the requirement that each of the four joint probabilities $P_{ij}(a_{ij},a_{ji})$ is bounded in $[0,1]$:

1. From Eq. (S3), we can isolate

$$P_{ij}(a_{ij}=1,a_{ji}=1) = p_{ij}p_{ji} + \rho_{ij}\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})} \ , \tag{S4}$$

which will be bounded in $[0,1]$ if

$$-\frac{p_{ij}p_{ji}}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \leq \rho_{ij} \leq \frac{1-p_{ij}p_{ji}}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \ . \tag{S5}$$

2. Combining Eqs. (S2a) and (S4), we can isolate

$$P_{ij}(a_{ij}=1,a_{ji}=0) = p_{ij} - P(a_{ij}=1,a_{ji}=1) = p_{ij}(1-p_{ji}) - \rho_{ij}\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})} \tag{S6}$$

which will be bounded in $[0,1]$ if

$$\frac{p_{ij}(1-p_{ji})-1}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \leq \rho_{ij} \leq \frac{p_{ij}(1-p_{ji})}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \ . \tag{S7}$$

3. Combining Eqs. (S2b) and (S4), we can isolate

$$P_{ij}(a_{ij}=0,a_{ji}=1) = p_{ji} - P_{ij}(a_{ij}=1,a_{ji}=1) = p_{ji}(1-p_{ij}) - \rho_{ij}\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})} \tag{S8}$$

which will be bounded in $[0,1]$ if

$$\frac{p_{ji}(1-p_{ij})-1}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \leq \rho_{ij} \leq \frac{p_{ji}(1-p_{ij})}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \ . \tag{S9}$$

4. Combining Eqs. (S1), (S4), (S6) and (S8), we can isolate

$$\begin{aligned} P_{ij}(a_{ij}=0,a_{ji}=0) &= 1 - P_{ij}(a_{ij}=1,a_{ji}=0) - P_{ij}(a_{ij}=0,a_{ji}=1) - P_{ij}(a_{ij}=1,a_{ji}=1) \\ &= (1-p_{ji})(1-p_{ij}) + \rho_{ij}\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})} \end{aligned} \tag{S10}$$

which will be bounded in $[0,1]$ if

$$-\frac{(1-p_{ji})(1-p_{ij})}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \leq \rho_{ij} \leq \frac{1-(1-p_{ji})(1-p_{ij})}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \ . \tag{S11}$$

The lower and upper bounds for ρ_{ij} are therefore

$$\begin{aligned} \rho_{ij}^{\min} &= \frac{1}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \max\left\{ -p_{ij}p_{ji},\ p_{ij}(1-p_{ji})-1,\ p_{ji}(1-p_{ij})-1,\ -(1-p_{ji})(1-p_{ij}) \right\} \\ &= \begin{cases} -\dfrac{p_{ij}p_{ji}}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} & \text{if } p_{ij}+p_{ji} < 1 \\ -\dfrac{(1-p_{ji})(1-p_{ij})}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} & \text{if } p_{ij}+p_{ji} > 1 \end{cases} \end{aligned} \tag{S12}$$

and

$$\begin{aligned} \rho_{ij}^{\max} &= \frac{1}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} \min\left\{ 1-p_{ij}p_{ji},\ p_{ij}(1-p_{ji}),\ p_{ji}(1-p_{ij}),\ 1-(1-p_{ji})(1-p_{ij}) \right\} \\ &= \begin{cases} \dfrac{p_{ij}(1-p_{ji})}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} & \text{if } p_{ij} < p_{ji} \\ \dfrac{p_{ji}(1-p_{ij})}{\sqrt{p_{ij}(1-p_{ij})p_{ji}(1-p_{ji})}} & \text{if } p_{ij} > p_{ji} \ . \end{cases} \end{aligned} \tag{S13}$$

To control the level of reciprocity, we introduce a parameter $\nu \in [-1,1]$ controlling ρ_{ij} such that $\rho_{ij}^{\min} \leq \rho_{ij} \leq \rho_{ij}^{\max}$

$$\rho_{ij} = \begin{cases} |\nu|\rho_{ij}^{\min} & \text{if } -1 \leq \nu \leq 0 \\ |\nu|\rho_{ij}^{\max} & \text{if } \quad 0 \leq \nu \leq 1 \end{cases} \ . \tag{S14}$$

Substituting Eq. (S14) into Eq. (S4) allows us to isolate

$$P_{ij}(a_{ij}=1,a_{ji}=1) = \begin{cases} (1+\nu)p_{ij}p_{ji} - \nu(p_{ij}+p_{ji}-1)H(p_{ij}+p_{ji}-1) & -1 \leq \nu \leq 0 \\ \\ (1-\nu)p_{ij}p_{ji} + \nu \min\left\{p_{ij}, p_{ji}\right\} & 0 \leq \nu \leq 1 \end{cases} , \tag{S15}$$

where $H(\cdot)$ is the Heaviside step function. This last equation alongside Eqs. (S1) and (S2) complete the approach for controlling reciprocity, whose level is tuned by the parameter ν.

S.II. ANALYSIS OF THE DIRECTED-RECIPROCAL $\mathbb{S}^1$ MODEL

A. Description of the model

We consider N nodes positioned on a circle of radius $R = N/2\pi$, thus setting the density of nodes to 1 without loss of generality. Each node i is independently and identically assigned an angular position θ_i and a pair of *hidden* degrees, κ_i^- and κ_i^+ which, as shown below, are related to their in- and out-degree, respectively. The angular positions are scattered on the circle according to the uniform probability density function (pdf)

$$\varphi(\theta) = \frac{1}{2\pi} \ . \tag{S16}$$

The hidden degrees are also assigned randomly according to the joint pdf $\rho(\kappa^-, \kappa^+)$, whose only constraint is

$$\iint \kappa^- \rho(\kappa^-, \kappa^+) d\kappa^- d\kappa^+ = \langle \kappa^- \rangle \equiv \langle \kappa \rangle \tag{S17a}$$

$$\iint \kappa^+ \rho(\kappa^-, \kappa^+) d\kappa^- d\kappa^+ = \langle \kappa^+ \rangle \equiv \langle \kappa \rangle \ . \tag{S17b}$$

A directed link exists from node i to node j with probability

$$P(a_{ij} = 1 | \kappa_i^+, \kappa_j^-, \Delta\theta_{ij}) = \frac{1}{1 + \chi_{ij}^\beta} \qquad \text{with} \quad \chi_{ij} = \frac{R\Delta\theta_{ij}}{\mu\kappa_i^+\kappa_j^-} = \frac{N\Delta\theta_{ij}}{2\pi\mu\kappa_i^+\kappa_j^-} \tag{S18}$$

where $\Delta\theta_{ij} = \Delta\theta_{ji} = \pi - |\pi - |\theta_i - \theta_j||$ is the minimal angular distance between nodes i and j, and where $\mu > 0$ and $\beta > 1$ are parameters of the model. Note that we will omit writing explicitly the dependency over β, μ and N for brevity. Note also that $\varphi(\theta) = \frac{1}{2\pi}$ implies that the pdf for $\Delta\theta_{ij}$ is simply $1/\pi$.

Two connection events may either be independent or *conditionally* independent. To see this, let us consider the following two entries of the adjacency matrix a_{ij} and a_{lk} with the associated probabilities of connection

$$P(a_{ij} = 1 | \kappa_i^+, \kappa_j^-, \Delta\theta_{ij}) = \frac{1}{1 + \left[\frac{N\Delta\theta_{ij}}{2\pi\mu\kappa_i^+\kappa_j^-}\right]^\beta} \ , \qquad \text{and} \qquad P(a_{lk} = 1 | \kappa_l^+, \kappa_k^-, \Delta\theta_{lk}) = \frac{1}{1 + \left[\frac{N\Delta\theta_{lk}}{2\pi\mu\kappa_l^+\kappa_k^-}\right]^\beta} \ ,$$

where we assume that $i \neq j$ and $l \neq k$ (i.e., no self loops). We distinguish several scenarios:

1. If $i = l$ and $j = k$, then the two connection events are trivially the same.
2. If $i = l$ and $j \neq k$ ($j = k$ and $i \neq l$), then the two links are outgoing (incoming) links from (on) the same node i (j) and both depend on the parameter κ_i^+ (κ_j^-). They are therefore conditionally independent.
3. If $i = k$ and $j \neq l$ ($j = l$ and $i \neq k$) then one link is outgoing from node i (j) and the other is incoming to node i (j). The two connection events will conditionally independent if and only if κ_i^{out} and κ_i^{in} (κ_j^{out} and κ_j^{in}) are correlated. Otherwise, the two connection events are independent.
4. If $i = k$ and $j = l$, then the two links are in opposite direction between the same two nodes. Both connection events depend on the angular separation $\Delta\theta_{ij}$ between the two nodes, and are therefore conditionally independent. The two connection events could be even further correlated if κ_i^{out} and κ_i^{in} (or equivalently κ_j^{out} and κ_j^{in}) are also correlated.
5. If i, j, l and k all take distinct values, then the two connection events are independent.

B. Out-degree of nodes

Let us first consider N nodes, each of which has been assigned an angular position θ, a hidden in-degree κ^- and a hidden out-degree κ^+. The sequence of angular positions, noted $\boldsymbol{\theta} \equiv \{\theta_1, \ldots, \theta_N\}$ is distributed according to the pdf $\prod_{i=1}^N \varphi(\theta_i) = (2\pi)^{-N}$, and the hidden degrees sequence, noted $\kappa \equiv \{\kappa_1^-, \kappa_1^+, \ldots, \kappa_N^-, \kappa_N^+\}$ is distributed according to the pdf $\prod_{i=1}^{N}\rho(\kappa_i^-,\kappa_i^+)$. The connection probability given by Eq. (S18) alongside the sequences $\boldsymbol{\theta}$ and $\boldsymbol{\kappa}$ define a random network ensemble in which node i has a out-degree equal to k_i^+ with probability $P_i^+(k_i^+|\boldsymbol{\theta},\boldsymbol{\kappa})$. The associated probability generating function (pgf) is defined as

$$H_i^+(z|\boldsymbol{\kappa},\boldsymbol{\theta}) = \sum_{k_i^+=0}^{N-1} P_i^+(k_i^+|\boldsymbol{\theta},\boldsymbol{\kappa}) z^{k_i^+} = \prod_{\substack{j=1\\ j\neq i}}^{N}\left[1 - P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij}) + zP(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij})\right] , \tag{S19}$$

where we used the fact that the existence of each link is *conditionally* independent from the existence of the others (i.e. they are independent events given the hidden variables θ_i and κ_i^+). General expressions for the expected out-degree of node i, its variance and for the ensemble average out-degree are respectively

$$\left\langle k_i^+ \middle| \boldsymbol{\kappa},\boldsymbol{\theta}\right\rangle = \left.\frac{\partial H_i^+(z|\boldsymbol{\kappa},\boldsymbol{\theta})}{\partial z}\right|_{z=1} = \sum_{\substack{j=1\\ j\neq i}}^{N} P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij}) , \tag{S20}$$

$$\begin{aligned}\mathrm{Var}\left[k_i^+ \middle| \boldsymbol{\kappa},\boldsymbol{\theta}\right] &= \left.\frac{\partial^2 H_i^+(z|\boldsymbol{\kappa},\boldsymbol{\theta})}{\partial z^2}\right|_{z=1} + \left.\frac{\partial H_i^+(z|\boldsymbol{\kappa},\boldsymbol{\theta})}{\partial z}\right|_{z=1} - \left[\left.\frac{\partial H_i^+(z|\boldsymbol{\kappa},\boldsymbol{\theta})}{\partial z}\right|_{z=1}\right]^2 \\ &= \sum_{\substack{j=1\\ j\neq i}}^{N} P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij})[1 - P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij})] ,\end{aligned} \tag{S21}$$

and

$$\left\langle k^+ \middle| \boldsymbol{\kappa},\boldsymbol{\theta}\right\rangle = \frac{1}{N}\sum_{i=1}^{N}\left\langle k_i^+ \middle| \boldsymbol{\kappa},\boldsymbol{\theta}\right\rangle = \frac{1}{N}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij}) . \tag{S22}$$

Let us now zoom out of the random network ensemble defined by specific sequences $\boldsymbol{\theta}$ and $\boldsymbol{\kappa}$ to focus instead on the random network ensemble defined by the pdfs $\varphi(\theta)$ and $\rho(\kappa^-,\kappa^+)$ (i.e. any sequences $\boldsymbol{\theta}$ of length N and $\boldsymbol{\kappa}$ of length $2N$ drawn from their respective pdf). Averaging over all angular positions (or, equivalently, over all angular distances), the expected probability for a link to exist from node i to node j in the network ensemble becomes

$$\begin{aligned}\left\langle a_{ij} \middle| \kappa_i^+,\kappa_j^-\right\rangle &= \frac{1}{\pi}\int_0^{\pi} P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij}) d\Delta\theta_{ij} \\ &= \frac{1}{\pi}\int_0^{\pi}\frac{1}{1+\chi_{ij}^{\beta}} d\Delta\theta_{ij} \\ &= \frac{2\mu\kappa_i^+\kappa_j^-}{N}\int_0^{\frac{N}{2\mu\kappa_i^+\kappa_j^-}}\frac{1}{1+\chi_{ij}^{\beta}} d\chi_{ij} \\ &= {}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-\left[\frac{N}{2\mu\kappa_i^+\kappa_j^-}\right]^{\beta}\right) && \text{(S23a)} \\ &\simeq \frac{2\pi\mu\kappa_i^+\kappa_j^-}{\beta N\sin(\pi/\beta)} \\ &= \frac{\kappa_i^+\kappa_j^-}{N\langle\kappa\rangle} , && \text{(S23b)}\end{aligned}$$

where $\simeq$ denotes an approximation that becomes exact in the limit $N/(\kappa_i^+\kappa_j^-)\to\infty$ [see Eqs. (S95) and (S104)], and where we set $\mu = \frac{\beta\sin(\pi/\beta)}{2\pi\langle\kappa\rangle}$ in the last equality. Averaging Eq. (S19) over every possible sequence $\boldsymbol{\theta}$ and $\boldsymbol{\kappa}\backslash\{\kappa_i^+\}$ then yields [note that $H_i^+(z|\boldsymbol{\kappa},\boldsymbol{\theta})$ does not depend on κ_i^-]

$$\begin{aligned}
H_i^+(z|\kappa_i^+) &= \int \cdots \int H_i^+(z|\boldsymbol{\kappa},\boldsymbol{\theta}) \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{\pi} d\Delta\theta_{ij} \rho(\kappa_j^-,\kappa_j^+) d\kappa_j^- d\kappa_j^+ \\
&= \prod_{\substack{j=1\\ j\neq i}}^{N} \Big[\iiint \Big[1 - P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij}) \\
&\qquad + zP(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij}) \Big] \frac{1}{\pi} d\Delta\theta_{ij} \rho(\kappa_j^-,\kappa_j^+) d\kappa_j^- d\kappa_j^+ \Big] \\
&= \prod_{\substack{j=1\\ j\neq i}}^{N} \Big[1 - \left\langle a_{i\bullet} \middle| \kappa_i^+ \right\rangle + z \left\langle a_{i\bullet} \middle| \kappa_i^+ \right\rangle \Big] \\
&= \Big[1 - \left\langle a_{i\bullet} \middle| \kappa_i^+ \right\rangle + z \left\langle a_{i\bullet} \middle| \kappa_i^+ \right\rangle \Big]^{N-1} ,
\end{aligned} \tag{S24}$$

where

$$\left\langle a_{i\bullet} \middle| \kappa_i^+ \right\rangle = \iint \left\langle a_{ij} \middle| \kappa_i^+, \kappa_j^- \right\rangle \rho(\kappa_j^-,\kappa_j^+) d\kappa_j^- d\kappa_j^+ \simeq \frac{2\pi\mu\langle\kappa\rangle\kappa_i^+}{\beta N \sin(\pi/\beta)} = \frac{\kappa_i^+}{N} \tag{S25}$$

is the average probability for the existence of any outgoing link from node i. From Eq. (S24), we conclude that the out-degree of node i in the random networks ensemble will be distributed according to a binomial distribution with average

$$\left\langle k_i^+ \middle| \kappa_i^+ \right\rangle = (N-1) \left\langle a_{i\bullet} \middle| \kappa_i^+ \right\rangle \simeq \frac{2\pi\mu\langle\kappa\rangle\kappa_i^+}{\beta \sin(\pi/\beta)} = \kappa_i^+ , \tag{S26}$$

and variance

$$\mathrm{Var}\left[k_i^+ \middle| \kappa_i^+ \right] = (N-1) \left\langle a_{i\bullet} \middle| \kappa_i^+ \right\rangle \left(1 - \left\langle a_{i\bullet} \middle| \kappa_i^+ \right\rangle \right) \simeq \frac{2\pi\mu\langle\kappa\rangle\kappa_i^+}{\beta \sin(\pi/\beta)} = \kappa_i^+ . \tag{S27}$$

Finally, the average out-degree in the ensemble of random networks is

$$\left\langle k^+ \right\rangle = \iint \left\langle k_i^+ \middle| \kappa_i^+ \right\rangle \rho(\kappa_i^-,\kappa_i^+) d\kappa_i^- d\kappa_i^+ \simeq \frac{2\pi\mu\langle\kappa\rangle^2}{\beta \sin(\pi/\beta)} = \langle\kappa\rangle . \tag{S28}$$

As $N/(\kappa^+\kappa^-) \to \infty$, the relative fluctuations around the expected out-degree, $\sqrt{\mathrm{Var}\left[k_i^+ \middle| \kappa_i^+ \right]} / \left\langle k_i^+ \middle| \kappa_i^+ \right\rangle$, will fall as $1/\sqrt{\kappa_i^+}$ and will become negligible for high out-degree nodes. The binomial distribution obtained in Eq. (S24) can therefore be approximated by a Poisson distribution in this limit

$$H_i^+(z|\kappa_i^+) = \sum_{k_i^+=0}^{N-1} P_i^+(k_i^+|\kappa_i^+) z^{k_i^+} \simeq \Big[1 + (z-1) \left\langle a_{i\bullet} \middle| \kappa_i^+ \right\rangle \Big]^{N-1} = \Big[1 + (z-1)\frac{\kappa_i^+}{N} \Big]^{N-1} \simeq \sum_{k_i^+=0}^{\infty} \frac{[\kappa_i^+]^{k_i^+} \mathrm{e}^{-\kappa_i^+}}{k_i^+!} z^{k_i^+} , \tag{S29}$$

where we identify

$$P_i^+(k_i^+|\kappa_i^+) \simeq \frac{[\kappa_i^+]^{k_i^+} \mathrm{e}^{-\kappa_i^+}}{k_i^+!} \tag{S30}$$

as the probability for node i with hidden out-degree κ_i^+ to have a degree equal to k_i^+.

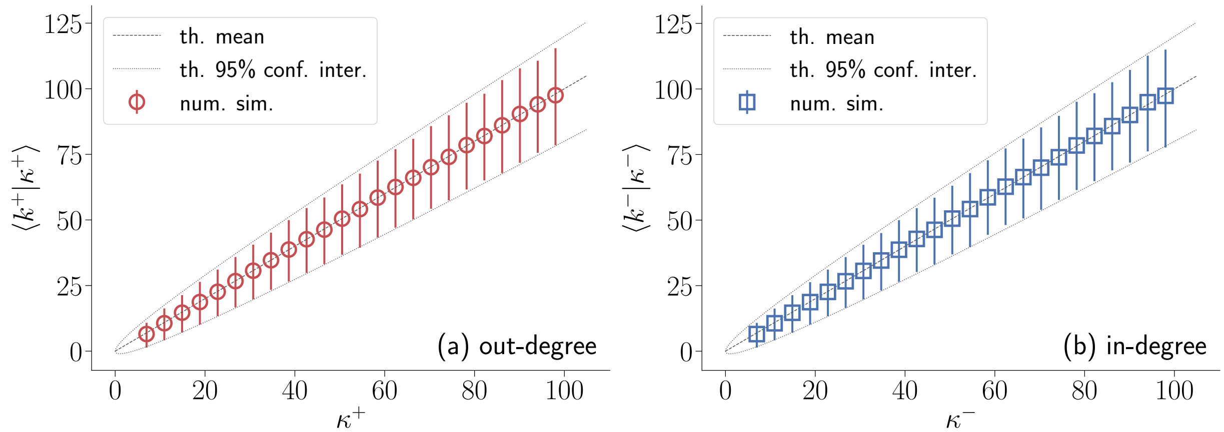

FIG. S1. **Validation of Eqs. (S26), (S27), (S32) and (S33) using numerical simulations.** Both κ^- and κ^+ were independently and identically drawn from the pdf $\rho(\kappa) \propto \kappa^{-2.5}$ with $5 < \kappa < 100$. Symbols show $\langle k^-|\kappa^-\rangle$ and $\langle k^+|\kappa^+\rangle$ estimated from 100 random synthetic networks with $N = 25000$. Only a fraction of the symbols are shown to avoid cluttering the plot. Error bars show the estimated 95% confidence interval.

C. In-degree of nodes

Repeating the same steps from the previous section yields an expression similar to Eq. (S25) for the average probability for the existence of any incoming link into node i

$$\langle a_{\bullet i}|\kappa_i^-\rangle = \iint \langle a_{ji}|\kappa_j^+, \kappa_i^-\rangle\, \rho(\kappa_j^-, \kappa_j^+) d\kappa_j^- d\kappa_j^+ \simeq \frac{2\pi\mu\langle\kappa\rangle\kappa_i^-}{\beta N \sin(\pi/\beta)} = \frac{\kappa_i^-}{N} \tag{S31}$$

an expression similar to Eq. (S26) for the expected in-degree of nodes

$$\langle k_i^-|\kappa_i^-\rangle \simeq \frac{2\pi\mu\langle\kappa\rangle\kappa_i^-}{\beta \sin(\pi/\beta)} = \kappa_i^- \ , \tag{S32}$$

and variance

$$\mathrm{Var}\left[k_i^-|\kappa_i^-\right] \simeq \frac{2\pi\mu\langle\kappa\rangle\kappa_i^-}{\beta \sin(\pi/\beta)} = \kappa_i^- \ , \tag{S33}$$

as well as an expression similar to Eq. (S28) for the ensemble average in-degree

$$\langle k^-\rangle \simeq \frac{2\pi\mu\langle\kappa\rangle^2}{\beta \sin(\pi/\beta)} = \langle\kappa\rangle \ . \tag{S34}$$

We also find that the probability for node i with hidden in-degree κ_i^- to have a degree equal to k_i^- to be

$$P_i^-(k_i^-|\kappa_i^-) \simeq \frac{[\kappa_i^-]^{k_i^-} \mathrm{e}^{-\kappa_i^-}}{k_i^-!} \ , \tag{S35}$$

similarly to Eq. (S30).

D. Joint in-/out-degree distribution

Since the existence of links is conditionally independent given the values of the hidden in- and out-degrees, the joint in-/out-degree distribution is

$$\begin{aligned} P(k^-, k^+) &= \iint P^-(k^-|\kappa^-) P^+(k^+|\kappa^+) \rho(\kappa^-, \kappa^+) d\kappa^- d\kappa^+ \\ &\simeq \iint \frac{[\kappa^-]^{k^-} \mathrm{e}^{-\kappa^-}}{k^-!} \frac{[\kappa^+]^{k^+} \mathrm{e}^{-\kappa^+}}{k^+!} \rho(\kappa^-, \kappa^+) d\kappa^- d\kappa^+ \ . \end{aligned} \tag{S36}$$

Hence, the in-degree and out-degree distributions are prescribed by their corresponding marginal pdf of $\rho(\kappa^-, \kappa^+)$ as

$$P^-(k^-) = \sum_{k^+=0}^{\infty} P(k^-, k^+) \simeq \int \frac{e^{-\kappa^-}[\kappa^-]^{k^-}}{k^-!} \left[\int \rho(\kappa^-, \kappa^+) d\kappa^+\right] d\kappa^- \tag{S37a}$$

$$P^+(k^+) = \sum_{k^-=0}^{\infty} P(k^-, k^+) \simeq \int \frac{e^{-\kappa^+}[\kappa^+]^{k^+}}{k^+!} \left[\int \rho(\kappa^-, \kappa^+) d\kappa^-\right] d\kappa^+ \ , \tag{S37b}$$

and the correlations between k^- and k^+ are governed by the correlations between κ^- and κ^+ encoded in $\rho(k^-, \kappa^+)$.

E. Reciprocal degree of nodes

Let us denote the probability for a reciprocal link to exist between nodes i and j by

$$P(a_{ij} = 1, a_{ji} = 1 | \kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) \ , \tag{S38}$$

with the additional assumption that this connection probability is symmetrical, i.e.

$$P(a_{ij} = 1, a_{ji} = 1 | \kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) = P(a_{ji} = 1, a_{ij} = 1 | \kappa_j^-, \kappa_j^+, \kappa_i^-, \kappa_i^+, \Delta\theta_{ji}) \ . \tag{S39}$$

Following similar steps to that in Sec. S.II B, we define the pgf associated with the reciprocal degree of node i given the sequences $\boldsymbol{\theta}$ and $\boldsymbol{\kappa}$ as

$$\begin{aligned} H_i^{\leftrightarrow}(z|\boldsymbol{\kappa}, \boldsymbol{\theta}) &= \sum_{k_i^{\leftrightarrow}=0}^{N-1} P_i^{\leftrightarrow}(k_i^{\leftrightarrow}|\boldsymbol{\theta}, \boldsymbol{\kappa}) z^{k_i^{\leftrightarrow}} \\ &= \prod_{\substack{j=1 \\ j\neq i}}^{N} \left[1 - P(a_{ij} = 1, a_{ji} = 1 | \kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) + zP(a_{ij} = 1, a_{ji} = 1 | \kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij})\right] \ . \end{aligned} \tag{S40}$$

Hence, general expressions for the expected reciprocal degree of node i and for the ensemble average reciprocal degree are respectively

$$\langle k_i^{\leftrightarrow} | \boldsymbol{\kappa}, \boldsymbol{\theta} \rangle = \left.\frac{\partial H_i^{\leftrightarrow}(z|\boldsymbol{\kappa}, \boldsymbol{\theta})}{\partial z}\right|_{z=1} = \sum_{\substack{j=1 \\ j\neq i}}^{N} P(a_{ij} = 1, a_{ji} = 1 | \kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) \ , \tag{S41}$$

and

$$\begin{aligned} \langle k^{\leftrightarrow} | \boldsymbol{\kappa}, \boldsymbol{\theta} \rangle &= \frac{1}{N} \sum_{i=1}^{N} \langle k_i^{\leftrightarrow} | \boldsymbol{\kappa}, \boldsymbol{\theta} \rangle \\ &= \frac{1}{N} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} P(a_{ij} = 1, a_{ji} = 1 | \kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) \\ &= \frac{2}{N} \sum_{i=1}^{N} \sum_{j=i+1}^{N} P(a_{ij} = 1, a_{ji} = 1 | \kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) \ . \end{aligned} \tag{S42}$$

Averaging $H_i^{\leftrightarrow}(z|\boldsymbol{\kappa},\boldsymbol{\theta})$ over every possible sequence $\boldsymbol{\theta}$ and $\boldsymbol{\kappa}\backslash\{\kappa_i^-,\kappa_i^+\}$ yields

$$\begin{aligned}
H_i^{\leftrightarrow}(z|\kappa_i^-,\kappa_i^+) &= \int\cdots\int_0^\pi H_i^{\leftrightarrow}(z|\boldsymbol{\kappa},\boldsymbol{\theta})\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{\pi}d\Delta\theta_{ij}\rho(\kappa_j^-,\kappa_j^+)d\kappa_j^-d\kappa_j^+ \\
&= \prod_{\substack{j=1\\ j\neq i}}^{N}\left[\iiint_0^\pi\Big[1-P(a_{ij}=1,a_{ji}=1|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\Delta\theta_{ij})\right. \\
&\qquad\left. + zP(a_{ij}=1,a_{ji}=1|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\Delta\theta_{ij})\Big]\frac{1}{\pi}d\Delta\theta_{ij}\rho(\kappa_j^-,\kappa_j^+)d\kappa_j^-d\kappa_j^+\right] \\
&= \prod_{\substack{j=1\\ j\neq i}}^{N}\left[\iint\Big[1-\left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+\right\rangle\right. \\
&\qquad\left. + z\left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+\right\rangle\Big]\rho(\kappa_j^-,\kappa_j^+)d\kappa_j^-d\kappa_j^+\right] \\
&= \prod_{\substack{j=1\\ j\neq i}}^{N}\Big[1-\left\langle a_{i\bullet}a_{\bullet i}\middle|\kappa_i^-,\kappa_i^+\right\rangle + z\left\langle a_{i\bullet}a_{\bullet i}\middle|\kappa_i^-,\kappa_i^+\right\rangle\Big] \\
&= \Big[1-\left\langle a_{i\bullet}a_{\bullet i}\middle|\kappa_i^-,\kappa_i^+\right\rangle + z\left\langle a_{i\bullet}a_{\bullet i}\middle|\kappa_i^-,\kappa_i^+\right\rangle\Big]^{N-1}\ ,
\end{aligned} \tag{S43}$$

where

$$\left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+\right\rangle = \int_0^\pi P(a_{ij}=1,a_{ji}=1|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\Delta\theta_{ij})\frac{1}{\pi}d\Delta\theta_{ij} \tag{S44}$$

and

$$\left\langle a_{i\bullet}a_{\bullet i}\middle|\kappa_i^-,\kappa_i^+\right\rangle = \iint\left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+\right\rangle\rho(\kappa_j^-,\kappa_j^+)d\kappa_j^-d\kappa_j^+\ . \tag{S45}$$

The expected reciprocal degree of node i is then

$$\left\langle k_i^{\leftrightarrow}\middle|\kappa_i^-,\kappa_i^+\right\rangle = (N-1)\left\langle a_{i\bullet}a_{\bullet i}\middle|\kappa_i^-,\kappa_i^+\right\rangle\ . \tag{S46}$$

The ensemble average expected reciprocal degree is

$$\langle k^{\leftrightarrow}\rangle = \iint\left\langle k_i^{\leftrightarrow}\middle|\kappa_i^-,\kappa_i^+\right\rangle\rho(\kappa_i^-,\kappa_i^+)d\kappa_i^-d\kappa_i^+\ . \tag{S47}$$

F. Reciprocity

We are now in a position to combine the results from Sec. S.I with those from the previous subsections to study the reciprocity in the networks generated by the directed-reciprocal $\mathbb{S}^1$ model. The reciprocity is defined as

$$r = \frac{L^{\leftrightarrow}}{L}\ , \tag{S48}$$

where L is the number of links, and $L^{\leftrightarrow}$ is the number of reciprocal links. Note that for r to be such that $0\leq r\leq 1$, each reciprocal connection (e.g. when two nodes are connected by two links in the opposite direction) must contribute 2 to $L^{\leftrightarrow}$. Hence $L^{\leftrightarrow}$ is an even number. Averaging Eq. (S48) over all possible angular positions $\boldsymbol{\theta}$ and hidden in/out-degrees $\boldsymbol{\kappa}$, we get

$$\langle r\rangle = \left\langle\frac{L^{\leftrightarrow}}{L}\right\rangle \approx \frac{\langle L^{\leftrightarrow}\rangle}{\langle L\rangle} = \frac{N\langle k^{\leftrightarrow}\rangle}{N\langle k^+\rangle} = \begin{cases}(1+\nu)\langle r|\nu=0\rangle - \nu\langle r|\nu=-1\rangle & -1\leq\nu\leq 0\\ (1-\nu)\langle r|\nu=0\rangle + \nu\langle r|\nu=1\rangle & 0\leq\nu\leq 1\end{cases}\ , \tag{S49}$$

where we used Eqs. (S44)–(S47), and where we defined the following quantities.

1. $\langle r|\nu=1\rangle$ is the expected reciprocity when $\nu=1$

$$\begin{aligned}\langle r|\nu=1\rangle &= \frac{\langle k^{\leftrightarrow}|\nu=1\rangle}{\langle k^{+}\rangle} \\ &= \frac{N-1}{\langle k^{+}\rangle}\iiiint \left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\nu=1\right\rangle \\ &\qquad\qquad \times\rho(\kappa_i^-,\kappa_i^+)\rho(\kappa_j^-,\kappa_j^+)d\kappa_i^- d\kappa_i^+ d\kappa_j^- d\kappa_j^+ \end{aligned} \tag{S50}$$

with $\left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\nu=1\right\rangle$ being the expected reciprocal connection probability, Eq. (S44), when $\nu=1$

$$\begin{aligned}\left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\nu=1\right\rangle &= \frac{1}{\pi}\int_0^{\pi}\min\left\{P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij}),P(a_{ji}=1|\kappa_j^+,\kappa_i^-,\Delta\theta_{ij})\right\}d\Delta\theta_{ij} \\ &= \frac{1}{\pi}\int_0^{\pi}\min\left\{\frac{1}{1+\chi_{ij}^{\beta}},\frac{1}{1+\chi_{ji}^{\beta}}\right\}d\Delta\theta_{ij} \\ &= H(1-\xi_{ij})\frac{1}{\pi}\int_0^{\pi}\frac{1}{1+\chi_{ij}^{\beta}}d\Delta\theta_{ij}+H(\xi_{ij}-1)\frac{1}{\pi}\int_0^{\pi}\frac{1}{1+\chi_{ji}^{\beta}}d\Delta\theta_{ij} \\ &= H(1-\xi_{ij})\frac{2\mu\kappa_i^+\kappa_j^-}{N}\int_0^{\frac{N}{2\mu\kappa_i^+\kappa_j^-}}\frac{1}{1+\chi_{ij}^{\beta}}d\chi_{ij} \\ &\qquad + H(\xi_{ij}-1)\frac{2\mu\kappa_j^+\kappa_i^-}{N}\int_0^{\frac{N}{2\mu\kappa_j^+\kappa_i^-}}\frac{1}{1+\chi_{ji}^{\beta}}d\chi_{ij} \\ &= H(1-\xi_{ij})\,{}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-\left[\frac{N}{2\mu\kappa_i^+\kappa_j^-}\right]^{\beta}\right) \\ &\qquad + H(\xi_{ij}-1)\,{}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-\left[\frac{N}{2\mu\kappa_j^+\kappa_i^-}\right]^{\beta}\right) \qquad\qquad (\text{S51}) \\ &\simeq H(1-\xi_{ij})\frac{2\pi\mu\kappa_i^+\kappa_j^-}{\beta N\sin(\pi/\beta)}+H(\xi_{ij}-1)\frac{2\pi\mu\kappa_j^+\kappa_i^-}{\beta N\sin(\pi/\beta)} \\ &= H(1-\xi_{ij})\frac{\kappa_i^+\kappa_j^-}{N\langle\kappa\rangle}+H(\xi_{ij}-1)\frac{\kappa_j^+\kappa_i^-}{N\langle\kappa\rangle}\ , \end{aligned} \tag{S52}$$

where we used Eqs. (S95) and (S104), and where we set $\mu=\dfrac{\beta\sin(\pi/\beta)}{2\pi\langle\kappa\rangle}$ and defined $\xi_{ij}=\frac{\kappa_i^+}{\kappa_i^-}\frac{\kappa_j^-}{\kappa_j^+}$.

2. $\langle r|\nu=0\rangle$ is the expected reciprocity when $\nu=0$

$$\begin{aligned}\langle r|\nu=0\rangle &= \frac{\langle k^{\leftrightarrow}|\nu=0\rangle}{\langle k^{+}\rangle} \\ &= \frac{N-1}{\langle k^{+}\rangle}\iiiint \left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\nu=0\right\rangle \\ &\qquad\qquad \times\rho(\kappa_i^-,\kappa_i^+)\rho(\kappa_j^-,\kappa_j^+)d\kappa_i^- d\kappa_i^+ d\kappa_j^- d\kappa_j^+ \end{aligned} \tag{S53}$$

with

$$\begin{aligned}
\left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\nu=0,\xi_{ij}=1\right\rangle &= \frac{1}{\pi}\int_0^\pi P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij})P(a_{ji}=1|\kappa_j^+,\kappa_i^-,\Delta\theta_{ij})d\Delta\theta_{ij} \\
&= \frac{1}{\pi}\int_0^\pi \frac{1}{1+\chi_{ij}^\beta}\frac{1}{1+\chi_{ji}^\beta}d\Delta\theta_{ij} \\
&= \frac{1}{\pi}\int_0^\pi \frac{1}{\left(1+\chi_{ij}^\beta\right)^2}d\Delta\theta_{ij} \\
&= \frac{2\mu\kappa_i^+\kappa_j^-}{N}\int_0^{\frac{N}{2\mu\kappa_i^+\kappa_j^-}} \frac{1}{\left(1+\chi_{ij}^\beta\right)^2}d\chi_{ij} \\
&= {}_2F_1\left(2,\frac{1}{\beta};1+\frac{1}{\beta};-\left[\frac{N}{2\mu\kappa_i^+\kappa_j^-}\right]^\beta\right) \qquad \text{(S54)}\\
&\simeq \frac{2\pi\mu\kappa_i^+\kappa_j^-}{\beta N\sin(\pi/\beta)}\left(1-\frac{1}{\beta}\right) \\
&= \frac{\kappa_i^+\kappa_j^-}{N\langle\kappa\rangle}\left(1-\frac{1}{\beta}\right) \qquad \text{(S55)}
\end{aligned}$$

when $\xi_{ij}=\frac{\kappa_i^+}{\kappa_i^-}\frac{\kappa_j^-}{\kappa_j^+}=1$, and

$$\begin{aligned}
\left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\nu=0,\xi_{ij}\neq 1\right\rangle &= \frac{1}{\pi}\int_0^\pi P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij})P(a_{ji}=1|\kappa_j^+,\kappa_i^-,\Delta\theta_{ij})d\Delta\theta_{ij} \\
&= \frac{1}{\pi}\int_0^\pi \frac{1}{1+\chi_{ij}^\beta}\frac{1}{1+\chi_{ji}^\beta}d\Delta\theta_{ij} \\
&= \frac{1}{\pi}\int_0^\pi \frac{1}{1+\chi_{ij}^\beta}\frac{1}{1+\xi_{ij}^\beta\chi_{ij}^\beta}d\Delta\theta_{ij} \\
&= \frac{2\mu\kappa_i^+\kappa_j^-}{N}\int_0^{\frac{N}{2\mu\kappa_i^+\kappa_j^-}} \frac{1}{1+\chi_{ij}^\beta}\frac{1}{1+\xi_{ij}^\beta\chi_{ij}^\beta}d\chi_{ij} \\
&= \frac{1}{1-\xi_{ij}^\beta}\,{}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-\left[\frac{N}{2\mu\kappa_i^+\kappa_j^-}\right]^\beta\right) \\
&\qquad -\frac{\xi_{ij}^\beta}{1-\xi_{ij}^\beta}\,{}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-\left[\frac{N}{2\mu\kappa_j^+\kappa_i^-}\right]^\beta\right) \qquad \text{(S56)}\\
&\simeq \frac{2\pi\mu\kappa_i^+\kappa_j^-}{\beta N\sin(\pi/\beta)}\frac{1-\xi_{ij}^{\beta-1}}{1-\xi_{ij}^\beta} \\
&= \frac{\kappa_i^+\kappa_j^-}{N\langle\kappa\rangle}\frac{1-\xi_{ij}^{\beta-1}}{1-\xi_{ij}^\beta} \qquad \text{(S57)}
\end{aligned}$$

otherwise. In the last two equations, we again set $\mu=\frac{\beta\sin(\pi/\beta)}{2\pi\langle\kappa\rangle}$, and used Eqs. (S101), (S104) and (S105).

3. $\langle r|\nu=-1\rangle$ is the expected reciprocity when $\nu=-1$

$$\begin{aligned}
\langle r|\nu=-1\rangle &= \frac{\langle k^\leftrightarrow|\nu=-1\rangle}{\langle k^+\rangle} \\
&= \frac{N-1}{\langle k^+\rangle}\iiiint \left\langle a_{ij}a_{ji}\middle|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\nu=-1\right\rangle \\
&\qquad \times\rho(\kappa_i^-,\kappa_i^+)\rho(\kappa_j^-,\kappa_j^+)d\kappa_i^-d\kappa_i^+d\kappa_j^-d\kappa_j^+ \qquad \text{(S58)}
\end{aligned}$$

with

$$\begin{aligned}
\big\langle a_{ij}a_{ji}\big|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\nu=-1\big\rangle &= \frac{1}{\pi}\int_0^{\pi}\Big[P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij})+P(a_{ji}=1|\kappa_j^+,\kappa_i^-,\Delta\theta_{ij})-1\Big] \\
&\qquad \times H\Big(P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij})+P(a_{ji}=1|\kappa_j^+,\kappa_i^-,\Delta\theta_{ij})-1\Big)d\Delta\theta_{ij} \\
&= \frac{1}{\pi}\int_0^{\Delta\theta_{ij}^{\mathrm{c}}}\Big[P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij})+P(a_{ji}=1|\kappa_j^+,\kappa_i^-,\Delta\theta_{ij})-1\Big]d\Delta\theta_{ij} \\
&= \frac{1}{\pi}\int_0^{\Delta\theta_{ij}^{\mathrm{c}}}\frac{d\Delta\theta_{ij}}{1+\chi_{ij}^{\beta}}+\frac{1}{\pi}\int_0^{\Delta\theta_{ij}^{\mathrm{c}}}\frac{d\Delta\theta_{ij}}{1+\chi_{ji}^{\beta}}-\frac{1}{\pi}\int_0^{\Delta\theta_{ij}^{\mathrm{c}}}d\Delta\theta_{ij} \\
&= \frac{2\mu\kappa_i^+\kappa_j^-}{N}\int_0^{\frac{N\Delta\theta_{ij}^{\mathrm{c}}}{2\pi\mu\kappa_i^+\kappa_j^-}}\frac{d\chi_{ij}}{1+\chi_{ij}^{\beta}}+\frac{2\mu\kappa_j^+\kappa_i^-}{N}\int_0^{\frac{N\Delta\theta_{ij}^{\mathrm{c}}}{2\pi\mu\kappa_j^+\kappa_i^-}}\frac{d\chi_{ji}}{1+\chi_{ji}^{\beta}}-\frac{\Delta\theta_{ij}^{\mathrm{c}}}{\pi} \\
&= \frac{\Delta\theta_{ij}^{\mathrm{c}}}{\pi}\left[{}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-\left[\frac{N\Delta\theta_{ij}^{\mathrm{c}}}{2\pi\mu\kappa_i^+\kappa_j^-}\right]^{\beta}\right)\right. \\
&\qquad\left.+\,{}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-\left[\frac{N\Delta\theta_{ij}^{\mathrm{c}}}{2\pi\mu\kappa_j^+\kappa_i^-}\right]^{\beta}\right)-1\right]
\end{aligned} \tag{S59}$$

where we used Eq. (S95), and where $\Delta\theta_{ij}^{\mathrm{c}}$ is the solution of

$$P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij}^{\mathrm{c}})+P(a_{ji}=1|\kappa_j^+,\kappa_i^-,\Delta\theta_{ij}^{\mathrm{c}})=1\ . \tag{S60}$$

To explore the limit $N\to\infty$ such that $N/(\kappa_i^+\kappa_j^-)\to\infty$ and $N/(\kappa_j^+\kappa_i^-)\to\infty$, we note that

$$\begin{aligned}
1 &= P(a_{ij}=1|\kappa_i^+,\kappa_j^-,\Delta\theta_{ij}^{\mathrm{c}})+P(a_{ji}=1|\kappa_j^+,\kappa_i^-,\Delta\theta_{ij}^{\mathrm{c}}) \\
&= \frac{1}{1+\left[\frac{N\Delta\theta_{ij}^{\mathrm{c}}}{2\pi\mu\kappa_i^+\kappa_j^-}\right]^{\beta}}+\frac{1}{1+\left[\frac{N\Delta\theta_{ij}^{\mathrm{c}}}{2\pi\mu\kappa_j^+\kappa_i^-}\right]^{\beta}} \\
&\simeq \frac{[2\pi\mu\kappa_i^+\kappa_j^-]^{\beta}}{[N\Delta\theta_{ij}^{\mathrm{c}}]^{\beta}}+\frac{[2\pi\mu\kappa_j^+\kappa_i^-]^{\beta}}{[N\Delta\theta_{ij}^{\mathrm{c}}]^{\beta}}\ ,
\end{aligned} \tag{S61}$$

and thus

$$\Delta\theta_{ij}^{\mathrm{c}}\simeq\frac{2\pi\mu}{N}\Big[[\kappa_i^+\kappa_j^-]^{\beta}+[\kappa_j^+\kappa_i^-]^{\beta}\Big]^{\frac{1}{\beta}}=\frac{2\pi\mu\kappa_i^+\kappa_j^-}{N}\Big[1+\xi_{ij}^{-\beta}\Big]^{\frac{1}{\beta}}\ . \tag{S62}$$

Equation (S59) then becomes

$$\begin{aligned}
\big\langle a_{ij}a_{ji}\big|\kappa_i^-,\kappa_i^+,\kappa_j^-,\kappa_j^+,\nu=-1\big\rangle &\simeq \frac{2\mu\kappa_i^+\kappa_j^-}{N}\Big[1+\xi_{ij}^{-\beta}\Big]^{\frac{1}{\beta}}\left[{}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-1-\xi_{ij}^{-\beta}\right)\right. \\
&\qquad\left.+\,{}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-1-\xi_{ij}^{\beta}\right)-1\right] \\
&= \frac{\kappa_i^+\kappa_j^-}{N\langle\kappa\rangle}\frac{\sin(\pi/\beta)}{\pi/\beta}\Big[1+\xi_{ij}^{-\beta}\Big]^{\frac{1}{\beta}}\left[{}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-1-\xi_{ij}^{-\beta}\right)\right. \\
&\qquad\left.+\,{}_2F_1\left(1,\frac{1}{\beta};1+\frac{1}{\beta};-1-\xi_{ij}^{\beta}\right)-1\right]\ ,
\end{aligned} \tag{S63}$$

where we set $\mu=\dfrac{\beta\sin(\pi/\beta)}{2\pi\langle\kappa\rangle}$.

S.III. NETWORK DATASETS

The network datasets used in the article have been made publicly available by the original authors and were downloaded from The Netzschleuder network catalogue and repository (https://networks.skewed.de). The name of each dataset is listed below:

7th_graders	add_health_comm15	add_health_comm27
add_health_comm28	add_health_comm33	add_health_comm40
add_health_comm41	add_health_comm50	add_health_comm61
add_health_comm62	add_health_comm68	add_health_comm73
add_health_comm75	add_health_comm79	add_health_comm81
add_health_comm83	add_health_comm84	advogato
anybeat	bison	bitcoin_alpha
bitcoin_trust	caida_as_20040105	caida_as_20040202
caida_as_20040301	caida_as_20040405	caida_as_20040503
caida_as_20040607	caida_as_20040705	caida_as_20040802
caida_as_20040906	caida_as_20041004	caida_as_20041101
caida_as_20041206	caida_as_20050103	caida_as_20050207
caida_as_20050307	caida_as_20050404	caida_as_20050502
caida_as_20050606	caida_as_20050704	caida_as_20050801
caida_as_20050905	caida_as_20051003	caida_as_20051107
caida_as_20051205	caida_as_20060102	caida_as_20060109
caida_as_20060116	caida_as_20060123	caida_as_20060130
caida_as_20060206	caida_as_20060213	caida_as_20060220
caida_as_20060227	caida_as_20060306	caida_as_20060313
caida_as_20060320	caida_as_20060327	caida_as_20060403
caida_as_20060410	caida_as_20060417	caida_as_20060424
caida_as_20060501	caida_as_20060508	caida_as_20060515
caida_as_20060522	caida_as_20060529	caida_as_20060605
caida_as_20060612	caida_as_20060619	caida_as_20060626
caida_as_20060703	caida_as_20060710	caida_as_20060717
caida_as_20060724	caida_as_20060731	caida_as_20060807
caida_as_20060814	caida_as_20060821	caida_as_20060828
caida_as_20060904	caida_as_20060911	caida_as_20060918
caida_as_20060925	caida_as_20061002	caida_as_20061009
caida_as_20061016	caida_as_20061023	caida_as_20061030
caida_as_20061106	caida_as_20061113	caida_as_20061120
caida_as_20061127	caida_as_20061204	caida_as_20061211
caida_as_20061218	caida_as_20061225	caida_as_20070101
caida_as_20070108	caida_as_20070115	caida_as_20070122
caida_as_20070129	caida_as_20070205	caida_as_20070212
caida_as_20070219	caida_as_20070226	caida_as_20070305
caida_as_20070312	caida_as_20070423	caida_as_20070917
cattle	celegans_2019_hermaphrodite_chemical	celegans_2019_hermaphrodite_chemical_corrected
celegans_2019_hermaphrodite_chemical_synapse	celegans_2019_male_chemical	celegans_2019_male_chemical_corrected
celegans_2019_male_chemical_synapse	celegansneural	chess
chicago_road	cintestinalis	college_freshmen
copenhagen_calls	copenhagen_sms	cora
dblp_cite	dutch_school_klas12b-net-1	dutch_school_klas12b-net-2
dutch_school_klas12b-net-3	dutch_school_klas12b-net-3m	dutch_school_klas12b-net-4
dutch_school_klas12b-net-4m	dutch_school_klas12b-primary	ecoli_transcription_v1_0
ecoli_transcription_v1_1	email_company	faa_routes
fao_trade	foodweb_baywet	foodweb_little_rock
fresh_webs_AkatoreA	fresh_webs_AkatoreB	fresh_webs_Berwick
fresh_webs_Blackrock	fresh_webs_Broad	fresh_webs_Canton
fresh_webs_Catlins	fresh_webs_Coweeta1	fresh_webs_Coweeta17
fresh_webs_DempstersAu	fresh_webs_DempstersSp	fresh_webs_DempstersSu
fresh_webs_German	fresh_webs_Healy	fresh_webs_Kyeburn
fresh_webs_LilKyeburn	fresh_webs_Martins	fresh_webs_Narrowdale
fresh_webs_NorthCol	fresh_webs_Powder	fresh_webs_Stony
fresh_webs_SuttonAu	fresh_webs_SuttonSp	fresh_webs_SuttonSu
fresh_webs_Troy	fresh_webs_Venlaw	freshman_t0
freshman_t2	freshman_t3	freshman_t5

freshman_t6	freshmen_t0	freshmen_t2
freshmen_t3	freshmen_t5	freshmen_t6
genetic_multiplex_Arabidopsis	genetic_multiplex_Bos_Multiplex_Genetic	genetic_multiplex_Candida
genetic_multiplex_Celegans	genetic_multiplex_DanioRerio	genetic_multiplex_Drosophila
genetic_multiplex_Gallus	genetic_multiplex_HepatitusCVirus	genetic_multiplex_HumanHIV1
genetic_multiplex_HumanHerpes4	genetic_multiplex_Mus	genetic_multiplex_Oryctolagus
genetic_multiplex_Plasmodium	genetic_multiplex_Rattus	genetic_multiplex_Sacchpomb
genetic_multiplex_Xenopus	gnutella_04	gnutella_06
gnutella_08	gnutella_09	gnutella_25
hens	high_tech_company	highschool
inploid	interactome_figeys	interactome_stelzl
jdk	jung	law_firm
macaques	messal_shale	moreno_sheep
moreno_taro	openflights	packet_delays
physician_trust	polblogs	qa_user_mathoverflow_a2q
qa_user_mathoverflow_c2a	qa_user_mathoverflow_c2q	residence_hall
rhesus_monkey	sp_high_school_diaries	sp_high_school_survey
un_migrations	uni_email	us_agencies_alabama
us_agencies_alaska	us_agencies_arizona	us_agencies_arkansas
us_agencies_california	us_agencies_colorado	us_agencies_connecticut
us_agencies_delaware	us_agencies_florida	us_agencies_georgia
us_agencies_hawaii	us_agencies_idaho	us_agencies_illinois
us_agencies_indiana	us_agencies_iowa	us_agencies_kansas
us_agencies_kentucky	us_agencies_louisiana	us_agencies_maine
us_agencies_maryland	us_agencies_massachusetts	us_agencies_michigan
us_agencies_minnesota	us_agencies_mississippi	us_agencies_missouri
us_agencies_montana	us_agencies_nebraska	us_agencies_nevada
us_agencies_newhampshire	us_agencies_newjersey	us_agencies_newmexico
us_agencies_newyork	us_agencies_northcarolina	us_agencies_northdakota
us_agencies_ohio	us_agencies_oklahoma	us_agencies_oregon
us_agencies_pennsylvania	us_agencies_rhodeisland	us_agencies_southcarolina
us_agencies_southdakota	us_agencies_tennessee	us_agencies_texas
us_agencies_utah	us_agencies_vermont	us_agencies_virginia
us_agencies_washington	us_agencies_westvirginia	us_agencies_wisconsin
us_agencies_wyoming	us_air_traffic	webkb_webkb_cornell_link1
webkb_webkb_texas_link1	webkb_webkb_washington_link1	webkb_webkb_wisconsin_link1
wiki_talk_br	wiki_talk_cy	wiki_talk_eo
wiki_talk_gl	wiki_talk_ht	wiki_talk_nds
wiki_talk_oc	wikipedia_link_si	word_adjacency_darwin
word_adjacency_french	word_adjacency_japanese	word_adjacency_spanish
yeast_transcription		

S.IV. INFERENCE ALGORITHM

The inference algorithm used in the main text is an adaptation of the parameter inference procedure of the embedding algorithm introduced in Ref. [1]. Its objective is to infer the $2N+2$ parameters $\boldsymbol{\kappa} = \kappa_1^-, \kappa_1^+, \ldots, \kappa_N^-, \kappa_N^+$, β and ν so that the directed-reciprocal $\mathbb{S}^1$ model will reproduce, on average, the joint in/out-degree sequence, the reciprocity and the density of triangles of an original real directed network ($2N+2$ constraints).

Note that, contrary to the embedding algorithm introduced in Ref. [1], the inference algorithm does not aim to infer the angular positions, $\boldsymbol{\theta}$; the aforementioned $2N+2$ parameters are therefore inferred when averaging over all possible angular positions.

A. Inputs

The following $2N+2$ constraints are measured on an original real directed network and used as inputs for the inference algorithm.

1. The joint in/out-degree sequence $\boldsymbol{k} = \{k_1^-, k_1^+, \ldots, k_N^-, k_N^+\}$, where

$$k_i^- = |\partial_i^-| \tag{S64a}$$
$$k_i^+ = |\partial_i^+| \ , \tag{S64b}$$

   and where ∂_i^- (∂_i^+) is the set of in-neighbors (out-neighbors) of node i in the original real directed network.

2. The reciprocity r^{obs} computed as

$$r^{\text{obs}} = \frac{L^{\leftrightarrow}}{L} = \frac{\sum_{i=1}^N |\partial_i^- \cap \partial_i^+|}{\sum_{i=1}^N |\partial_i^+|} \ , \tag{S65}$$

   where $|\partial_i^- \cap \partial_i^+|$ counts the number of neighbors with which node i shares both possible directed links (i.e. reciprocal connection).

3. The density of triangles, $\bar{c}_{\text{obs}}$, as measured by the average undirected local clustering coefficient

$$\bar{c}_{\text{undir}}^{\text{obs}} = \frac{1}{N_{>1}} \sum_{i=1}^N c_i = \frac{1}{N_{>1}} \sum_{i=1}^N \frac{2T_i}{|\partial_i^- \cup \partial_i^+|\big(|\partial_i^- \cup \partial_i^+| - 1\big)} \tag{S66}$$

   where T_i is the number of triangles to which node i participates, where the quantity $|\partial_i^- \cup \partial_i^+|$ corresponds to the degree of node i in the undirected version of the network, and where $N_{>1}$ is the number of nodes for which $|\partial_i^- \cup \partial_i^+| > 1$. Note that we set $c_i = 0$ for the $N - N_{>1}$ nodes for which $|\partial_i^- \cup \partial_i^+| < 2$.

B. Inferring the hidden in/out-degrees

This subroutine assumes that a maximal deviation tolerance $\varepsilon_{\text{tol}}^{\max}$ and the parameter β have both been assigned some value (e.g. $\varepsilon_{\text{tol}}^{\max} = 0.01$), and uses

$$\mu = \frac{\beta \sin\left(\frac{\pi}{\beta}\right)}{2\pi\langle k^+\rangle} \tag{S67}$$

where $\langle k^+\rangle$ is the average out-degree (or equivalently average in-degree) in the original real directed network

$$\langle k^+\rangle = \frac{1}{N}\sum_{i=1}^N k_i^+ = \frac{1}{N}\sum_{i=1}^N k_i^- \ . \tag{S68}$$

1. *Initialize* the hidden in/out-degrees by setting $\kappa_i^- = k_i^-$ and $\kappa_i^+ = k_i^+$ for all $i = 1, \ldots, N$.

2. *Compute expected in/out-degrees* as

$$\langle k_i^- | \boldsymbol{\kappa} \rangle = \sum_{\substack{j=1 \\ j\neq i}}^{N} \langle a_{ji} | \kappa_j^+, \kappa_i^- \rangle = \sum_{\substack{j=1 \\ j\neq i}}^{N} {}_2F_1\left(1, \frac{1}{\beta}; 1+\frac{1}{\beta}; -\left[\frac{N}{2\mu\kappa_j^+\kappa_i^-}\right]^\beta\right) , \tag{S69a}$$

$$\langle k_i^+ | \boldsymbol{\kappa} \rangle = \sum_{\substack{j=1 \\ j\neq i}}^{N} \langle a_{ij} | \kappa_i^+, \kappa_j^- \rangle = \sum_{\substack{j=1 \\ j\neq i}}^{N} {}_2F_1\left(1, \frac{1}{\beta}; 1+\frac{1}{\beta}; -\left[\frac{N}{2\mu\kappa_i^+\kappa_j^-}\right]^\beta\right) , \tag{S69b}$$

for all $i = 1, \ldots, N$. Equations (S69a) and (S69b) are obtained by averaging Eq. (S20) (and its equivalent for in-degrees) over all angular positions, combined with Eq. (S23a).

3. *Compute the largest deviation*, $\varepsilon^{\max}$, between the expected in/out-degrees and the in/out-degrees in the original network as

$$\varepsilon^{\max} = \max\left\{ \max\left\{ \left| \langle k_i^- | \boldsymbol{\kappa} \rangle - k_i^- \right|, \left| \langle k_i^+ | \boldsymbol{\kappa} \rangle - k_i^+ \right| \right\} : i = 1, \ldots, N \right\} . \tag{S70}$$

The hidden in/out-degrees have converged to acceptable values if $\varepsilon^{\max} < \varepsilon^{\max}_{\mathrm{tol}}$ and we proceed to step 6. Otherwise, they require more refinement and we proceed to step 4.

4. *Update the hidden in/out-degrees* according to

$$\kappa_i^- \leftarrow \left| \kappa_i^- + \left[k_i^- - \langle k_i^- | \boldsymbol{\kappa} \rangle \right] u^- \right| , \tag{S71a}$$

$$\kappa_i^+ \leftarrow \left| \kappa_i^+ + \left[k_i^+ - \langle k_i^+ | \boldsymbol{\kappa} \rangle \right] u^+ \right| , \tag{S71b}$$

for all $i = 1, \ldots, N$, and where $u^- \sim \mathrm{Uniform}(0,1)$ and $u^+ \sim \mathrm{Uniform}(0,1)$. The random variables prevent the subroutine from getting trapped in a local minimum.

5. *Proceed to step 2* using the updated values for the hidden in/out-degrees.

6. *Compute the expected in-degree and out-degree* as

$$\langle k^- | \boldsymbol{\kappa} \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle k_i^- | \boldsymbol{\kappa} \rangle \tag{S72a}$$

$$\langle k^+ | \boldsymbol{\kappa} \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle k_i^+ | \boldsymbol{\kappa} \rangle . \tag{S72b}$$

Equations (S72a) and (S72b) are obtained by averaging Eq. (S22) (and its equivalent for in-degrees) over all angular positions. Note that $\langle k^- | \boldsymbol{\kappa} \rangle$ and $\langle k^+ | \boldsymbol{\kappa} \rangle$ will be equal up to the numerical error induced by $\varepsilon^{\max}_{\mathrm{tol}}$.

C. Inferring parameter ν

This subroutine assumes that the parameter β has been assigned some value, and uses the parameter μ, the hidden in/out-degrees, $\boldsymbol{\kappa} = \kappa_1^-, \kappa_1^+, \ldots, \kappa_N^-, \kappa_N^+$, as well as the expected out-degree, $\langle k^+ | \boldsymbol{\kappa} \rangle$, computed in Sec. S.IV B.

The expected reciprocity in the directed-reciprocal $\mathbb{S}^1$ model is computed as

$$\langle r | \boldsymbol{\kappa} \rangle = \left\langle \frac{L^{\leftrightarrow}}{L} \middle| \boldsymbol{\kappa} \right\rangle \approx \frac{\langle L^{\leftrightarrow} | \boldsymbol{\kappa} \rangle}{\langle L | \boldsymbol{\kappa} \rangle} = \frac{N \langle k^{\leftrightarrow} | \boldsymbol{\kappa} \rangle}{N \langle k^+ | \boldsymbol{\kappa} \rangle} , \tag{S73}$$

where $\langle k^+ | \boldsymbol{\kappa} \rangle$ is taken from Eq. (S72b) and $\langle k^{\leftrightarrow} | \boldsymbol{\kappa} \rangle$ is computed by averaging Eq. (S42) over all angular positions. Equation (S73) then takes a similar form as Eq. (S49) and becomes

$$\langle r | \boldsymbol{\kappa} \rangle \approx \begin{cases} (1+\nu) \langle r | \boldsymbol{\kappa}, \nu = 0 \rangle - \nu \langle r | \boldsymbol{\kappa}, \nu = -1 \rangle & \text{if } -1 \leq \nu \leq 0 \\ (1-\nu) \langle r | \boldsymbol{\kappa}, \nu = 0 \rangle + \nu \langle r | \boldsymbol{\kappa}, \nu = 1 \rangle & \text{if } \quad 0 \leq \nu \leq 1 \end{cases} , \tag{S74}$$

and the inferred value of ν is obtained such that $\langle r|\boldsymbol{\kappa}\rangle = r^{\rm obs}$. This subroutine computes $\langle r|\boldsymbol{\kappa}\rangle$ and ν via the following steps.

1. *Compute the expected reciprocity when* $\nu = 1$ using

$$\langle r|\boldsymbol{\kappa}, \nu{=}1\rangle = \frac{\langle k^{\leftrightarrow}|\boldsymbol{\kappa}, \nu{=}1\rangle}{\langle k^{+}|\boldsymbol{\kappa}\rangle} = \frac{2}{N\,\langle k^{+}|\boldsymbol{\kappa}\rangle} \sum_{i=1}^{N}\sum_{j=i+1}^{N} \big\langle a_{ij}a_{ji}\big|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \nu{=}1\big\rangle \tag{S75a}$$

where

$$\big\langle a_{ij}a_{ji}\big|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \nu{=}1\big\rangle = \begin{cases} {}_2F_1\left(1, \dfrac{1}{\beta}; 1+\dfrac{1}{\beta}; -\left[\dfrac{N}{2\mu\kappa_i^+\kappa_j^-}\right]^{\beta}\right) & \text{if } \xi_{ij} = \dfrac{\kappa_i^+}{\kappa_i^-}\dfrac{\kappa_j^-}{\kappa_j^+} < 1 \\[2ex] {}_2F_1\left(1, \dfrac{1}{\beta}; 1+\dfrac{1}{\beta}; -\left[\dfrac{N}{2\mu\kappa_j^+\kappa_i^-}\right]^{\beta}\right) & \text{if } \xi_{ij} = \dfrac{\kappa_i^+}{\kappa_i^-}\dfrac{\kappa_j^-}{\kappa_j^+} > 1 \end{cases} . \tag{S75b}$$

Equations (S75a) and (S75b) are obtained by averaging Eq. (S42) over all angular positions, combined with Eq. (S51).

2. *Compute the expected reciprocity when* $\nu = 0$ using

$$\langle r|\boldsymbol{\kappa}, \nu{=}0\rangle = \frac{\langle k^{\leftrightarrow}|\boldsymbol{\kappa}, \nu{=}0\rangle}{\langle k^{+}|\boldsymbol{\kappa}\rangle} = \frac{2}{N\,\langle k^{+}|\boldsymbol{\kappa}\rangle} \sum_{i=1}^{N}\sum_{j=i+1}^{N} \big\langle a_{ij}a_{ji}\big|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \nu{=}0\big\rangle \tag{S76a}$$

where

$$\big\langle a_{ij}a_{ji}\big|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \nu{=}0\big\rangle = \begin{cases} \dfrac{1}{1-\xi_{ij}^{\beta}}\,{}_2F_1\left(1, \dfrac{1}{\beta}; 1+\dfrac{1}{\beta}; -\left[\dfrac{N}{2\mu\kappa_i^+\kappa_j^-}\right]^{\beta}\right) & \\ \quad -\dfrac{\xi_{ij}^{\beta}}{1-\xi_{ij}^{\beta}}\,{}_2F_1\left(1, \dfrac{1}{\beta}; 1+\dfrac{1}{\beta}; -\left[\dfrac{N}{2\mu\kappa_j^+\kappa_i^-}\right]^{\beta}\right) & \text{if } \xi_{ij} = \dfrac{\kappa_i^+}{\kappa_i^-}\dfrac{\kappa_j^-}{\kappa_j^+} \neq 1 \\[2ex] {}_2F_1\left(2, \dfrac{1}{\beta}; 1+\dfrac{1}{\beta}; -\left[\dfrac{N}{2\mu\kappa_i^+\kappa_j^-}\right]^{\beta}\right) & \text{if } \xi_{ij} = \dfrac{\kappa_i^+}{\kappa_i^-}\dfrac{\kappa_j^-}{\kappa_j^+} = 1 \end{cases} . \tag{S76b}$$

Equations (S76a) and (S76b) are obtained by averaging Eq. (S42) over all angular positions, combined with Eqs. (S54) and (S56).

3. *Compute the expected reciprocity when* $\nu = -1$ using

$$\langle r|\boldsymbol{\kappa}, \nu{=}0\rangle = \frac{\langle k^{\leftrightarrow}|\boldsymbol{\kappa}, \nu{=}{-1}\rangle}{\langle k^{+}|\boldsymbol{\kappa}\rangle} = \frac{2}{N\,\langle k^{+}|\boldsymbol{\kappa}\rangle} \sum_{i=1}^{N}\sum_{j=i+1}^{N} \big\langle a_{ij}a_{ji}\big|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \nu{=}{-1}\big\rangle \tag{S77a}$$

where

$$\big\langle a_{ij}a_{ji}\big|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \nu{=}{-1}\big\rangle = \frac{\Delta\theta_{ij}^{\rm c}}{\pi}\left[{}_2F_1\left(1, \frac{1}{\beta}; 1+\frac{1}{\beta}; -\left[\frac{N\Delta\theta_{ij}^{\rm c}}{2\pi\mu\kappa_i^+\kappa_j^-}\right]^{\beta}\right) + {}_2F_1\left(1, \frac{1}{\beta}; 1+\frac{1}{\beta}; -\left[\frac{N\Delta\theta_{ij}^{\rm c}}{2\pi\mu\kappa_j^+\kappa_i^-}\right]^{\beta}\right) - 1\right] \tag{S77b}$$

and where $\Delta\theta_{ij}^{\rm c} \in [0,\pi]$ is the solution of

$$P(a_{ij}=1|\kappa_i^+, \kappa_j^-, \Delta\theta_{ij}^{\rm c}) + P(a_{ji}=1|\kappa_j^+, \kappa_i^-, \Delta\theta_{ij}^{\rm c}) = 1 \ . \tag{S77c}$$

Equations (S77a)–(S77c) are obtained by averaging Eq. (S42) over all angular positions, combined with Eq. (S59).

4. *Compute the inferred value of* ν according to

$$\nu = \begin{cases} \dfrac{r^{\rm obs} - \langle r|\boldsymbol{\kappa}, \nu\!=\!0\rangle}{\langle r|\boldsymbol{\kappa}, \nu\!=\!-1\rangle + \langle r|\boldsymbol{\kappa}, \nu\!=\!0\rangle} & \text{if } r^{\rm obs} < \langle r|\boldsymbol{\kappa}, \nu\!=\!0\rangle \\ \\ \dfrac{r^{\rm obs} - \langle r|\boldsymbol{\kappa}, \nu\!=\!0\rangle}{\langle r|\boldsymbol{\kappa}, \nu\!=\!1\rangle - \langle r|\boldsymbol{\kappa}, \nu\!=\!0\rangle} & \text{if } r^{\rm obs} > \langle r|\boldsymbol{\kappa}, \nu\!=\!0\rangle \end{cases} . \tag{S78}$$

D. Estimating the expected density of triangles

This subroutine assumes that the parameter β has been assigned some value, uses the parameter ν computed in Sec. S.IV C, and uses the parameter μ as well as the hidden in/out-degrees, $\boldsymbol{\kappa} = \kappa_1^-, \kappa_1^+, \ldots, \kappa_N^-, \kappa_N^+$ computed in Sec. S.IV B.

The density of triangles is quantified using the average *undirected* local clustering coefficient, that is the average local clustering coefficient measured on the *undirected projection* of the directed network. This projection is specified via its adjacency matrix, $\tilde{\mathbf{A}}$, whose elements are $\tilde{a}_{ij} = \max(a_{ij}, a_{ji})$. In other words, two nodes are connected in the projection if they are connected by at least one directed link, which occurs with probability

$$\begin{aligned} P(\tilde{a}_{ij} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) = P_{ij}(a_{ij} = 1, a_{ji} = 0|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) \\ + P_{ij}(a_{ij} = 0, a_{ji} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) \\ + P_{ij}(a_{ij} = 1, a_{ji} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) . \end{aligned} \tag{S79}$$

This last expression can be rewritten as

$$\begin{aligned} P(\tilde{a}_{ij} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) = P_{ij}(a_{ij} = 1|\kappa_i^+, \kappa_j^-, \Delta\theta_{ij}) \\ + P_{ij}(a_{ji} = 1|\kappa_i^-, \kappa_j^+, \Delta\theta_{ij}) \\ - P_{ij}(a_{ij} = 1, a_{ji} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) , \end{aligned} \tag{S80}$$

where the three probabilities on the right-hand side are obtained using Eqs. (S15) and (S18). Let us also introduce $P(\tilde{a}_{ij} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+)$ which corresponds to $P(\tilde{a}_{ij} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij})$ averaged over all possible angular positions

$$\begin{aligned} P(\tilde{a}_{ij} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+) &= \int P(\tilde{a}_{ij} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+) P(\Delta\theta_{ij}) d\Delta\theta_{ij} \\ &= \left\langle a_{ij}\middle|\kappa_i^+, \kappa_j^-\right\rangle + \left\langle a_{ji}\middle|\kappa_j^+, \kappa_i^-\right\rangle - \left\langle a_{ij}a_{ji}\middle|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+\right\rangle , \end{aligned} \tag{S81}$$

where $\left\langle a_{ij}\middle|\kappa_i^+, \kappa_j^-\right\rangle$ and $\left\langle a_{ji}\middle|\kappa_j^+, \kappa_i^-\right\rangle$ are computed using Eq. (S23a), and where

$$\left\langle a_{ij}a_{ji}\middle|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+\right\rangle = \begin{cases} (1+\nu)\left\langle a_{ij}a_{ji}\middle|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \nu\!=\!0\right\rangle \\ \qquad -\nu\left\langle a_{ij}a_{ji}\middle|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \nu\!=\!-1\right\rangle & -1 \le \nu \le 0 \\ \\ (1-\nu)\left\langle a_{ij}a_{ji}\middle|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \nu\!=\!0\right\rangle \\ \qquad +\nu\left\langle a_{ij}a_{ji}\middle|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \nu\!=\!1\right\rangle & 0 \le \nu \le 1 \end{cases} \tag{S82}$$

is computed using Eqs. (S51), (S54), (S56) and (S59).

From these quantities, we use Bayes theorem to define two probability distributions with which we estimate the expected density of triangles. The first one corresponds to the probability that neighbor j of node i has hidden degrees κ_j^-, κ_j^+ regardless of the angular distance

$$P(\kappa_j^-, \kappa_j^+|\tilde{a}_{ij} = 1, \kappa_i^-, \kappa_i^+) = \frac{P(\tilde{a}_{ij} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+) P(\kappa_j^-, \kappa_j^+)}{P(\tilde{a}_{i\bullet} = 1|\kappa_i^-, \kappa_i^+)} \tag{S83}$$

where $P(\tilde{a}_{i\bullet} = 1|\kappa_i^-, \kappa_i^+)$ is a normalization constant. The second distribution provides the probability that neighboring nodes i and j are at angular distance $\Delta\theta_{ij}$

$$P(\Delta\theta_{ij}|\tilde{a}_{ij} = 1, \kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+) = \frac{P(\tilde{a}_{ij} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+, \Delta\theta_{ij}) P(\Delta\theta_{ij})}{P(\tilde{a}_{ij} = 1|\kappa_i^-, \kappa_i^+, \kappa_j^-, \kappa_j^+)} . \tag{S84}$$

Recall that $P(\Delta\theta_{ij}) = 1/\pi$ in the directed-reciprocal $\mathbb{S}^1$ model.

With these quantities in hand, the expected density of triangles is estimated by computing

$$\bar{c}_{\mathrm{undir}} \approx \frac{1}{MN_{>1}} \sum_{i=1}^{N} \sum_{m=1}^{M} c_i^{(m)} \mathbb{1}_{\{k_i^- + k_i^+ > 1\}} \ , \tag{S85}$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function, M is the number of samples to be drawn for each node i, and where the m-th sample, $c_i^{(m)}$, is obtained with the following procedure.

1. *Pick the hidden degrees of two neighbors*, (κ_1^-, κ_1^+) and (κ_2^-, κ_2^+), by sampling Eq. (S83) twice.
2. *Pick the angular distance* between node i and nodes 1 and 2, $\Delta\theta_{i1}$ and $\Delta\theta_{i2}$, by sampling Eq. (S84) twice.
3. *Set the angular distance between nodes 1 and 2.* Since it is equally likely for nodes 1 and 2 to be "on the same side" or "on opposite sides" from node i, we set

$$\Delta\theta_{12} = \begin{cases} \min\left\{|\Delta\theta_{i1} + \Delta\theta_{i2}|, 2\pi - |\Delta\theta_{i1} + \Delta\theta_{i2}|\right\} & \text{with probability } 1/2 \\ \\ |\Delta\theta_{i1} - \Delta\theta_{i2}| & \text{with probability } 1/2 \end{cases} . \tag{S86}$$

4. *Compute the probability for nodes 1 and 2 to be connected* and set $c_i^{(m)} = P(\tilde{a}_{12} = 1 | \kappa_1^-, \kappa_1^+, \kappa_2^-, \kappa_2^+, \Delta\theta_{12})$.

Note: The average undirected local clustering coefficient is a convenient measure to estimate the value of β. However, because it does not fully embrace the direction of links, leading to an ambiguous definition of the degree used at the denominator (see Methods), the estimated value for β may require some manual adjustments for the model to accurately reproduce the number of triangles observed in the original network. See Fig. S2 for an illustration.

E. The algorithm

The algorithm assumes that a maximal deviation tolerance, η^{tol}, has been assigned, as well as defines $\bar{c}_{\mathrm{undir}}^{\min} = 0$, $\bar{c}_{\mathrm{undir}}^{\max} = 1$, $\beta^{\min} = 1$ and $\beta^{\max} = 25$.

1. *Set the initial value for the parameter* β as $\beta = 1 + u$ where $u \sim \mathrm{Uniform}(0, 1)$.
2. *Infer the hidden in/out-degrees* $\boldsymbol{\kappa} = \kappa_1^-, \kappa_1^+, \ldots, \kappa_N^-, \kappa_N^+$ by following the procedure explained in Sec. S.IV B.
3. *Infer the parameter* ν by following the procedure explained in Sec. S.IV C.
4. *Estimate the triangle density* $\bar{c}_{\mathrm{undir}}$ by following the procedure explained in Sec. S.IV D.
5. *Check for convergence* by checking if $|\bar{c}_{\mathrm{undir}} - \bar{c}_{\mathrm{undir}}^{\mathrm{obs}}| < \eta^{\mathrm{tol}}$, then all $2N + 2$ parameters have been estimated within the tolerance parameters. Otherwise, proceed to step 6.
6. *Update the value of the parameter* β (bisection method):
   (a) If $\bar{c}_{\mathrm{undir}} > \bar{c}_{\mathrm{undir}}^{\mathrm{obs}}$, then set $\beta^{\max} = \beta$, set $\bar{c}_{\mathrm{undir}}^{\max} = \bar{c}_{\mathrm{undir}}$ and proceed to step 6c.
   (b) If $\bar{c}_{\mathrm{undir}} < \bar{c}_{\mathrm{undir}}^{\mathrm{obs}}$, then set $\beta^{\min} = \beta$, set $\bar{c}_{\mathrm{undir}}^{\min} = \bar{c}_{\mathrm{undir}}$ and proceed to step 6c.
   (c) Update β to its new value according to

$$\beta = \beta^{\min} + (\beta^{\max} - \beta^{\min}) \frac{\bar{c}_{\mathrm{undir}}^{\mathrm{obs}} - \bar{c}_{\mathrm{undir}}^{\min}}{\bar{c}_{\mathrm{undir}}^{\max} - \bar{c}_{\mathrm{undir}}^{\min}} \ .$$

   (d) Proceed to step 2.

S.V. MODELING REAL DIRECTED COMPLEX NETWORKS

A. Reproducing clustering and reciprocity

Figures S2 and S3 compare the accuracy with which the directed-reciprocal $\mathbb{S}^1$ model reproduces the average undirected local clustering coefficient, the number of triangles, as well as the reciprocity.

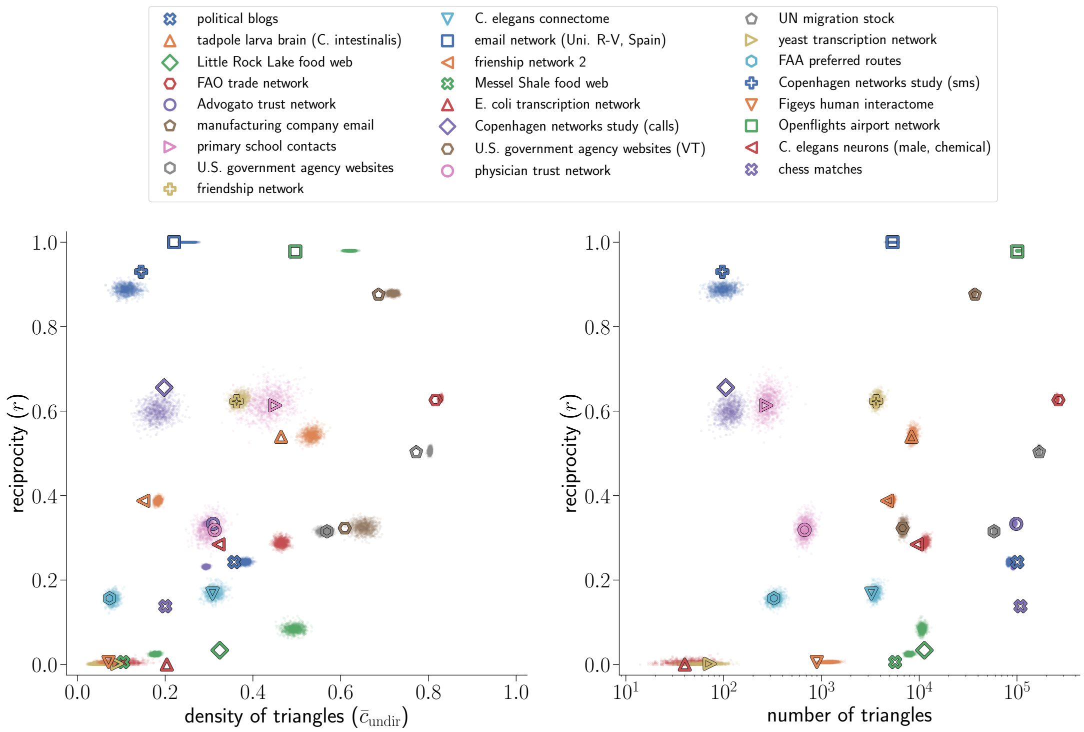

FIG. S2. Symbols represent the values measured on the original networks, and the small translucid circles show the same values measured on synthetic networks generated using the parameters inferred by the algorithm presented in Sec. S.IV (1000 network instances).

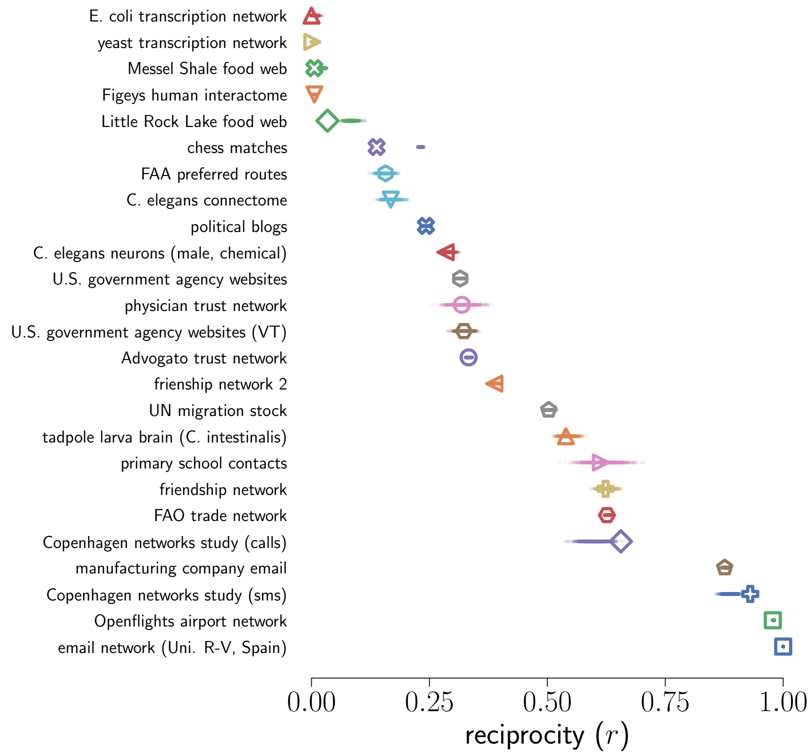

FIG. S3. Symbols represent the values measured on the original networks, and the small translucid circles show the same values measured on synthetic networks generated using the parameters inferred by the algorithm presented in Sec. S.IV (1000 network instances).

B. Additional triangle spectra

Figure S4 provides further examples of the capacity of the directed-reciprocal $\mathbb{S}^1$ model to reproduce the triangle spectra observed in various real directed networks. Table SI contains the parameters inferred for each network dataset.

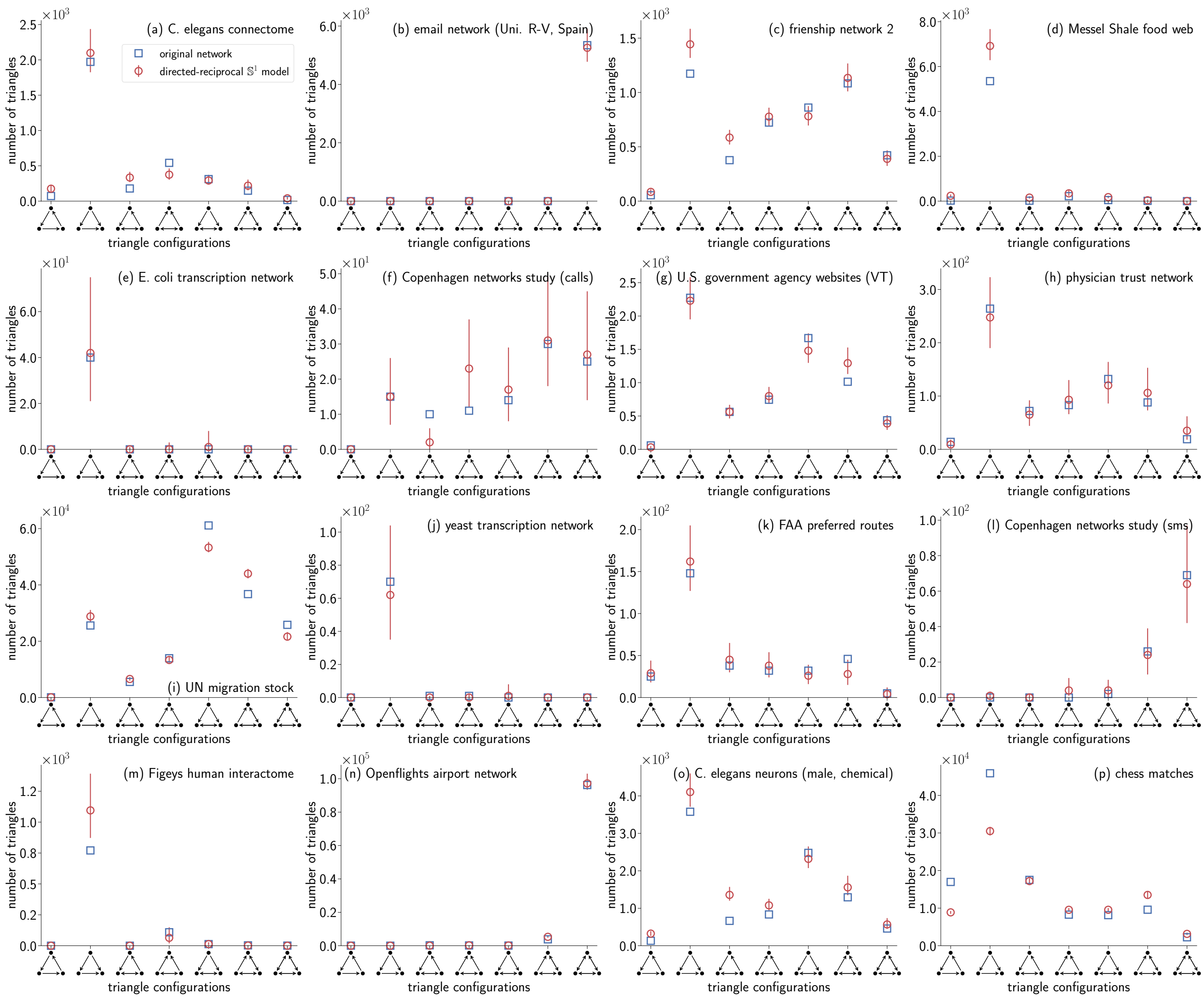

FIG. S4. **Reproducing triangle spectrum of real directed networks with the directed-reciprocal $\mathbb{S}^1$ model.** (a) Neural connections of the C. elegans nematode (dataset **celegansneural** [2, 3]). (b) Emails among members of a university (dataset **uni_email** [4]). (c) Friendships among high school students(dataset **add_health_comm50** [5]). (d) Messel Shale food web (dataset **messal_shale** [6]). (e) E. coli transcription network (dataset **ecoli_transcription_v1_0** [7]). (f) Social interactions among university students (dataset **copenhagen_calls** [8]). (g) Links between Vermont's government agencies websites (dataset: **us_agencies_vermont**) [9]). (h) Trust relationships among physicians (dataset **physician_trust** [10]). (i) Migration between countries (dataset **un_migrations** [11]). (j) Yeast transcription network (dataset **yeast_transcription** [12]). (k) Air traffic routes (dataset **faa_routes** [13]). (l) Social interactions among university students (dataset **copenhagen_sms** [8]). (m) Binding interactions between human proteins (dataset **interactome_figeys** [14]). (n) Regularly occurring flights among airports worldwide (dataset **openflights** [15]). (o) Networks among neurons of both the adult male and adult hermaphrodite worms C. elegans (dataset **celegans_2019_male_chemical** [16]). (p) Match outcomes between chess players (dataset **chess** [17]). Network datasets were downloaded from The Netzschleuder network catalogue and repository (https://networks.skewed.de). For each dataset, the parameters of the directed-reciprocal $\mathbb{S}^1$ model were adjusted using the inference procedure described in Sec. S.IV. Vertical lines show the estimated 95% confidence interval (2.5 and 97.5 percentiles).

TABLE SI. Parameters inferred for the network datasets used in Fig. 4 and Fig. S4.

Dataset	Figure	β	ν
political blogs	4(e)	1.75	0.43
tadpole larva brain (C. intestinalis)	4(f)	2.00	0.23
Little Rock Lake food web	4(g)	1.50	-1.00
FAO trade network	4(h)	1.75	0.34
Advogato trust network	4(i)	1.24	0.66
manufacturing company email	4(j)	1.50	1.00
primary school contacts	4(k)	2.43	0.88
U.S. government agency websites	4(l)	1.26	0.32
friendship network	4(m)	1.82	0.73
C. elegans connectome	S4(a)	1.28	-0.41
email network (Uni. R-V, Spain)	S4(b)	1.45	1.00
frienship network 2	S4(c)	1.40	0.48
Messel Shale food web	S4(d)	1.01	-0.91
E. coli transcription network	S4(e)	1.50	-0.02
Copenhagen networks study (calls)	S4(f)	1.50	1.00
U.S. government agency websites (VT)	S4(g)	2.00	0.11
physician trust network	S4(h)	1.67	0.27
UN migration stock	S4(i)	4.50	-0.73
yeast transcription network	S4(j)	1.20	0.02
FAA preferred routes	S4(k)	1.01	0.08
Copenhagen networks study (sms)	S4(l)	1.30	1.00
Figeys human interactome	S4(m)	1.01	1.00
Openflights airport network	S4(n)	3.25	1.00
C. elegans neurons (male, chemical)	S4(o)	2.00	-0.42
chess matches	S4(p)	1.30	-0.17

S.VI. USEFUL RESULTS INVOLVING THE HYPERGEOMETRIC FUNCTION

Letting $a, b \in \mathbb{C}$, $c \in \mathbb{C}\backslash\{0, -1, -2, -3, \ldots\}$ and $z \in \mathbb{Z}$, the hypergeometric function is defined by the Gauss series as [18]

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b+n)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{z^n}{n!} \tag{S87} $$

for $|z| < 1$ and elsewhere by analytic continuation. In what follows, we will use the following identity [19]

$$ \begin{aligned} {}_2F_1(a, b; c; z) = \frac{\pi\Gamma(c)}{\sin\pi(b-a)} \Bigg[& \frac{(-z)^{-a}}{\Gamma(c-a)\Gamma(b)\Gamma(a-b+1)} \; {}_2F_1\left(a, a-c+1, a-b+1; \frac{1}{z}\right) \\ & - \frac{(-z)^{-b}}{\Gamma(c-b)\Gamma(a)\Gamma(b-a+1)} \; {}_2F_1\left(b, b-c+1, b-a+1; \frac{1}{z}\right) \Bigg] \end{aligned} \tag{S88} $$

valid for $\arg(1-z) < \pi$, as well as [20]

$$ z \; {}_2F_1(a, b+1; c+1; z) = \frac{c}{b} \; {}_2F_1(a, b; c; z) - \frac{c}{b} \; {}_2F_1(a-1, b; c; z) \; . \tag{S89} $$

We will also need Euler's reflection formula [21]

$$ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{S90} $$

valid for $z \neq 0, \pm 1, \pm 2, \ldots$

We seek to evaluate the integral $\int \frac{dx}{1+x^\beta}$ for $x > 0$ and $\beta > 1$. To do so, we split the open interval $x > 0$ into two parts. First, we find for $0 < x < 1$

$$ \begin{aligned} \int \frac{dx}{1+x^\beta} &= \int \frac{1}{1-(-x^\beta)} dx \\ &= \int \sum_{n=0}^{\infty} \left(-x^\beta\right)^n dx \\ &= x \sum_{n=0}^{\infty} \frac{\left(-x^\beta\right)^n}{\beta n + 1} + C \\ &= x \sum_{n=0}^{\infty} \frac{\Gamma(1+n)}{n!\Gamma(1)} \frac{\Gamma(\frac{1}{\beta})}{\Gamma(\frac{1}{\beta})} \frac{\Gamma(\frac{1}{\beta}+n)}{\Gamma(\frac{1}{\beta}+n)} \frac{\frac{1}{\beta}}{\frac{1}{\beta}+n} \left(-x^\beta\right)^n + C_1 \\ &= x \sum_{n=0}^{\infty} \frac{\Gamma(1+n)}{\Gamma(1)} \frac{\Gamma(\frac{1}{\beta}+n)}{\Gamma(\frac{1}{\beta})} \frac{\Gamma(1+\frac{1}{\beta})}{\Gamma(1+\frac{1}{\beta}+n)} \frac{\left(-x^\beta\right)^n}{n!} + C_1 \\ &= x \; {}_2F_1\left(1, \frac{1}{\beta}; 1+\frac{1}{\beta}; -x^\beta\right) + C_1 \end{aligned} \tag{S91} $$

where $C_1 \in \mathbb{R}$. Second, we find for $x > 1$

$$\begin{aligned}
\int \frac{dx}{1+x^\beta} &= \int \frac{1}{x^\beta} \frac{1}{1-(-x^{-\beta})} dx \\
&= -\int \sum_{n=0}^{\infty} \left(-x^{-\beta}\right)^{n+1} dx \\
&= -x \sum_{n=0}^{\infty} \frac{\left(-x^{-\beta}\right)^{n+1}}{-\beta(n+1)+1} + C_2 \\
&= -x \sum_{m=1}^{\infty} \frac{\left(-x^{-\beta}\right)^{m}}{-\beta m+1} + C_2 \\
&= -x \sum_{m=1}^{\infty} \frac{\Gamma(1+m)}{m!\Gamma(1)} \frac{\Gamma(-\frac{1}{\beta})}{\Gamma(-\frac{1}{\beta})} \frac{\Gamma(-\frac{1}{\beta}+m)}{\Gamma(-\frac{1}{\beta}+m)} \frac{-\frac{1}{\beta}}{-\frac{1}{\beta}+m} \left(-x^{-\beta}\right)^{m} + C_2 \\
&= -x \sum_{m=1}^{\infty} \frac{\Gamma(1+m)}{\Gamma(1)} \frac{\Gamma(-\frac{1}{\beta}+m)}{\Gamma(-\frac{1}{\beta})} \frac{\Gamma(1-\frac{1}{\beta})}{\Gamma(1-\frac{1}{\beta}+m)} \frac{\left(-x^{-\beta}\right)^{m}}{m!} + C_2 \\
&= -x \left[{}_2F_1\left(1, -\frac{1}{\beta}; 1-\frac{1}{\beta}; -x^{-\beta}\right) - 1 \right] + C_2
\end{aligned} \tag{S92}$$

where $C_2 \in \mathbb{R}$. Combining Eqs. (S88) and (S90), we find

$${}_2F_1\left(1, -\frac{1}{\beta}; 1-\frac{1}{\beta}; -x^{-\beta}\right) = \frac{\frac{1}{\beta}}{1+\frac{1}{\beta}} x^\beta \; {}_2F_1\left(1, 1+\frac{1}{\beta}, 2+\frac{1}{\beta}; -x^\beta\right) + \frac{1}{\beta} \frac{\Gamma(-\frac{1}{\beta}-1)\Gamma(2+\frac{1}{\beta})}{x} . \tag{S93}$$

Using Eq. (S89), we find

$$x^\beta \; {}_2F_1\left(1, 1+\frac{1}{\beta}, 2+\frac{1}{\beta}; -x^\beta\right) = -\frac{1+\frac{1}{\beta}}{\frac{1}{\beta}} \; {}_2F_1\left(1, \frac{1}{\beta}; 1+\frac{1}{\beta}; -x^\beta\right) + \frac{1+\frac{1}{\beta}}{\frac{1}{\beta}} \tag{S94}$$

Combining Eqs. (S91)–(S94), we finally get

$$\int \frac{dx}{1+x^\beta} = x \; {}_2F_1\left(1, \frac{1}{\beta}; 1+\frac{1}{\beta}; -x^\beta\right) + C_3 \tag{S95}$$

for $x > 0$ and $\beta > 1$, and where $C_3 \in \mathbb{R}$.

We also seek to evaluate the integral $\int \frac{dx}{(1+x^\beta)^2}$ for $x > 0$ and $\beta > 1$. Again, we split the open interval $x > 0$ into two parts. First, we find for $0 < x < 1$

$$\begin{aligned}
\int \frac{dx}{(1+x^\beta)^2} &= \int \frac{d}{d(-x^\beta)} \frac{1}{1-(-x^\beta)} dx \\
&= \int \frac{d}{d(-x^\beta)} \sum_{n=0}^{\infty} \left(-x^\beta\right)^n dx \\
&= \int \sum_{n=1}^{\infty} n \left(-x^\beta\right)^{n-1} dx \\
&= x \sum_{m=0}^{\infty} \frac{(m+1)\left(-x^\beta\right)^m}{\beta m+1} + C_4 \\
&= x \sum_{m=0}^{\infty} (m+1) \frac{\Gamma(1+m)}{m!\Gamma(1)} \frac{\Gamma(\frac{1}{\beta})}{\Gamma(\frac{1}{\beta})} \frac{\Gamma(\frac{1}{\beta}+m)}{\Gamma(\frac{1}{\beta}+m)} \frac{\frac{1}{\beta}}{\frac{1}{\beta}+m} \left(-x^\beta\right)^m + C_4 \\
&= x \sum_{m=0}^{\infty} \frac{\Gamma(2+m)}{\Gamma(2)} \frac{\Gamma(\frac{1}{\beta}+m)}{\Gamma(\frac{1}{\beta})} \frac{\Gamma(1+\frac{1}{\beta})}{\Gamma(1+\frac{1}{\beta}+m)} \frac{\left(-x^\beta\right)^m}{m!} + C_4 \\
&= x \; {}_2F_1\left(2, \frac{1}{\beta}; 1+\frac{1}{\beta}; -x^\beta\right) + C_4
\end{aligned} \tag{S96}$$

where $C_4 \in \mathbb{R}$. Second, we find for $x > 1$

$$
\begin{aligned}
\int \frac{dx}{(1+x^\beta)^2} &= \int \frac{1}{x^{2\beta}} \frac{dx}{(1+x^{-\beta})^2} \\
&= \int x^{-2\beta} \frac{d}{d(-x^\beta)} \frac{1}{1-(-x^{-\beta})} dx \\
&= \int x^{-2\beta} \frac{d}{d(-x^{-\beta})} \sum_{n=0}^{\infty} \left(-x^{-\beta}\right)^n dx \\
&= \int \sum_{n=1}^{\infty} n \left(-x^{-\beta}\right)^{n+1} dx \\
&= x(-x^{-\beta}) \sum_{n=1}^{\infty} (-x^{-\beta}) \frac{d}{d(-x^{-\beta})} \frac{\left(-x^{-\beta}\right)^n}{-\beta(n+1)+1} + C_5 \\
&= x(-x^{-\beta})^2 \frac{d}{d(-x^{-\beta})} \sum_{n=1}^{\infty} \frac{\left(-x^{-\beta}\right)^n}{-\beta(n+1)+1} + C_5 \\
&= x(-x^{-\beta})^2 \frac{d}{d(-x^{-\beta})} \sum_{n=1}^{\infty} \frac{\Gamma(1+n)}{\Gamma(1)n!} \frac{-\frac{1}{\beta}}{1-\frac{1}{\beta}+n} \frac{1-\frac{1}{\beta}}{1-\frac{1}{\beta}} \frac{\Gamma(1-\frac{1}{\beta})}{\Gamma(1-\frac{1}{\beta})} \frac{\Gamma(1-\frac{1}{\beta}+n)}{\Gamma(1-\frac{1}{\beta}+n)} \left(-x^{-\beta}\right)^n + C_5 \\
&= x(-x^{-\beta})^2 \frac{-\frac{1}{\beta}}{1-\frac{1}{\beta}} \frac{d}{d(-x^{-\beta})} \sum_{n=1}^{\infty} \frac{\Gamma(1+n)}{\Gamma(1)} \frac{\Gamma(1-\frac{1}{\beta}+n)}{\Gamma(1-\frac{1}{\beta})} \frac{\Gamma(2-\frac{1}{\beta})}{\Gamma(2-\frac{1}{\beta}+n)} \frac{\left(-x^{-\beta}\right)^n}{n!} + C_5 \\
&= x(-x^{-\beta})^2 \frac{-\frac{1}{\beta}}{1-\frac{1}{\beta}} \frac{d}{d(-x^{-\beta})} \left[{}_2F_1\left(1, 1-\frac{1}{\beta}; 2-\frac{1}{\beta}; -x^{-\beta}\right) - 1 \right] + C_5 \\
&= \frac{-\frac{1}{\beta}}{2-\frac{1}{\beta}} x(-x^{-\beta})^2 \, {}_2F_1\left(2, 2-\frac{1}{\beta}; 3-\frac{1}{\beta}; -x^{-\beta}\right) + C_5
\end{aligned}
\tag{S97}
$$

where $C_5 \in \mathbb{R}$ and where we used the following identity [22] to obtain the last equality

$$
\frac{d}{dz} \, {}_2F_1(a,b;c;z) = \frac{ab}{c} \, {}_2F_1(a+1,b+1;c+1;z) \ . \tag{S98}
$$

Using Eqs. (S88) and (S90), Eq. (S97) becomes

$$
\int \frac{dx}{(1+x^\beta)^2} = x \, {}_2F_1\left(2, \frac{1}{\beta}; 1+\frac{1}{\beta}; -x^\beta\right) + (1-\tfrac{1}{\beta})\Gamma(1-\tfrac{1}{\beta})\Gamma(1+\tfrac{1}{\beta}) + C_5 \ , \tag{S99}
$$

which, combined with Eq. (S96), yields

$$
\int \frac{dx}{(1+x^\beta)^2} = x \, {}_2F_1\left(2, \frac{1}{\beta}; 1+\frac{1}{\beta}; -x^\beta\right) + C_6 \ , \tag{S100}
$$

for $x > 0$ and $\beta > 1$, and where $C_6 \in \mathbb{R}$.

We additionally seek to evaluate the following integral, which can be solved using Eqs. (S95) and (S100)

$$
\begin{aligned}
\int \frac{1}{1+x^\beta} \frac{1}{1+(\kappa x)^\beta} dx &= \frac{1}{1-\kappa^\beta} \int \frac{dx}{1+x^\beta} - \frac{\kappa^\beta}{1-\kappa^\beta} \int \frac{dx}{1+(\kappa x)^\beta} \\
&= \begin{cases} x \, {}_2F_1\left(2, \frac{1}{\beta}; 1+\frac{1}{\beta}; -x^\beta\right) + C_7 & \text{for } \kappa = 1 \\[2ex] \frac{x}{1-\kappa^\beta} \, {}_2F_1\left(1, \frac{1}{\beta}; 1+\frac{1}{\beta}; -x^\beta\right) \\ \qquad - \frac{x\kappa^\beta}{1-\kappa^\beta} \, {}_2F_1\left(1, \frac{1}{\beta}; 1+\frac{1}{\beta}; -(\kappa x)^\beta\right) + C_8 & \text{for } \kappa \neq 1 \end{cases}
\end{aligned}
\tag{S101}
$$

with $\kappa > 0$ and $\beta > 1$, $C_7, C_8 \in \mathbb{R}$ and $x > 0$.

Letting $d \in \{1, 2\}$, we use Eq. (S88) to write

$$z\ {}_2F_1\left(d, \frac{1}{\beta}; 1 + \frac{1}{\beta}; -z^\beta\right) = \frac{(-1)^d \pi}{\sin(\frac{\pi}{\beta})} \frac{\Gamma(1 + \frac{1}{\beta})}{\Gamma(1 + \frac{1}{\beta} - d)} \left[\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(d+n)\Gamma(d - \frac{1}{\beta} + n)}{\Gamma(d - \frac{1}{\beta})\Gamma(\frac{1}{\beta}) n! \Gamma(d - \frac{1}{\beta} + 1 + s)} z^{1-(n+d)\beta} - 1\right], \tag{S102}$$

which yields

$$\lim_{z\to\infty} z\ {}_2F_1\left(d, \frac{1}{\beta}; 1 + \frac{1}{\beta}; -z^\beta\right) = \frac{(-1)^{d+1} \pi}{\sin(\frac{\pi}{\beta})} \frac{\Gamma(1 + \frac{1}{\beta})}{\Gamma(1 + \frac{1}{\beta} - d)}, \tag{S103}$$

and more specifically

$$\lim_{z\to\infty} z\ {}_2F_1\left(1, \frac{1}{\beta}; 1 + \frac{1}{\beta}; -z^\beta\right) = \frac{\pi}{\beta} \frac{1}{\sin(\frac{\pi}{\beta})} \tag{S104}$$

and

$$\lim_{z\to\infty} z\ {}_2F_1\left(2, \frac{1}{\beta}; 1 + \frac{1}{\beta}; -z^\beta\right) = \frac{\pi(\beta - 1)}{\beta^2} \frac{1}{\sin(\frac{\pi}{\beta})}. \tag{S105}$$

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