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The multiple-resonator problem in a spherical GW antenna: its general solution and new interesting layouts

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Abstract. – This paper is intended to be a brief sketch of the main results which derive from a full theoretical implementation of the problem of how a set of mechanical resonators couples to an elastic sphere, excited by an incoming Gravitational Wave. The analysis reveals with remarkable transparency the general structure of the resonant mode splitting and coupling, and enables the discussion of interesting alternatives to the truncated icosahedron layout. A specific proposal for a rather complete monopole-quadrupole GW radiation detector is also made.

From a strictly theoretical point of view, a solid elastic sphere is the best resonant mass GW antenna one can possibly think of [1], yet practical difficulties have prevented to date the construction of such an antenna. One of the main difficulties has been the *readout system*, since it would require a *multiple transducer* set if the virtues of the sphere are to be exploited to satisfaction. Indeed, a *single* motion sensor would be insufficient to take adavantage of the perfect matching between the sphere's vibration modes and the GW radiation pattern. Long experience with cylindrical bars has now generated confidence that these problems can be solved, and so detailed analysis of the detector characteristics has become of the utmost interest. In recently published papers [2], [3], Johnson and Merkowitz have studied the problem of how a set of *resonant* transducers couples to the quadrupole modes of a solid elastic sphere. This study has led them to find a very interesting and highly symmetric resonator configuration, which they call TIGA, and which has been demonstrated experimentally on a small-scale prototype with remarkable success [4].

The theoretical model proposed by these authors is however restricted to the sphere's quadrupole modes. This actually obscures the nature of the approximations involved, but worse, has led Johnson and Merkovitz to inaccurate conclusions regarding the system dynamics for resonator configurations other than their TIGA. These, and other considerations of general theoretical character, have motivated us to attempt a more rigorous approach to the resonator problem. As a result, a theory of outstanding beauty has emerged, which fully displays the system response for *arbitrary* transducer layouts. Furthermore, when applied to the study of the frequency response of the TIGA prototype developed at LSU [4], our theory has been seen to make predictions which conform to the *actual* experimental measurements to *three and four decimal places* —a very remarkable achievement indeed in the reported [4] conditions.

In this letter we shall present the most relevant features of our analysis, along with a proposal for a *new* transducer layout, which will be shown to be an appealing alternative for a rather complete GW antenna. Full details will shortly be given elsewhere.

Following the notation of ref. [1], which will be used throughout, let a sphere of mss M, radius R, and density ρ, be endowed with a set of N identical resonators, each of mass $M_t \equiv \eta M$ and resonance frequency Ω, located at positions $\mathbf{x}_a$, $a=1,\ldots,N$, on the sphere's surface (so $|\mathbf{x}_a|=R$ for all a). We shall also assume that the resonators only couple to *radial motions*. If we call $\xi_a(t)$ the a-th resonator's displacement relative to the undeformed sphere's surface, and $\mathbf{n}_a$ the local outward normal, then the equations of motion for the whole system are

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} - \mu\nabla^2\mathbf{u} - (\lambda+\mu)\,\nabla(\nabla\cdot\mathbf{u}) = \tag{1a}$$
$$= \mathbf{f}(\mathbf{x},t) + \eta M\Omega^2\sum_{a=1}^{N}\delta^{(3)}(\mathbf{x}-\mathbf{x}_a)\,[\xi_a(t)-\mathbf{n}_a\cdot\mathbf{u}(\mathbf{x}_a,t)]\,\mathbf{n}_a\,,$$

$$\ddot{\xi}_a(t) = -\Omega^2\,[\xi_a(t)-\mathbf{n}_a\cdot\mathbf{u}(\mathbf{x}_a,t)]\ , \qquad a=1,\ldots,N \tag{1b}$$

and they are to be read as follows: the lhs of eq. (1a) contains the inertia and the *internal* strain forces in the solid, while its rhs contains the *external* forces; the latter are split into a *driving* force density $\mathbf{f}(\mathbf{x},t)$ acting on the sphere —such as a GW tide, a hammer stroke, or whatever—, *and* the action of the resonators on its surface. Equation (1b) is quite simply seen to describe the motion of harmonic oscillators, one of whose ends is rigidly linked to the sphere's surface. Equations (1) are completely general, and valid for any solid body and boundary conditions. Thus, although we shall exclusively concentrate in this paper on a perfect sphere, any deviations thereof can, and must, be handled by the same general system of equations.

The solution to (1) can be found in terms of Green-function expansions. The Green function for driving forces of the *separable* type

$$\mathbf{f}(\mathbf{x},t) = \sum_\alpha \mathbf{f}^{(\alpha)}(\mathbf{x})\,g^{(\alpha)}(t) \tag{2}$$

has been calculated in [1], and in particular for GW excitations, which happen to be of this form. The reader is referred to that paper for details. The *complete* solution to eqs. (1) is, however, not interesting, as the very purpose of attaching transducers is precisely to sense the sphere's motion by sampling it at a *finite* number of points on its surface. In other words, the *measurable* quantities

$$q_a(t) \equiv \xi_a(t) - \mathbf{n}_a\cdot\mathbf{u}(\mathbf{x}_a,t)\ , \qquad a=1,\ldots,N \tag{3}$$

are the only ones of our concern. It turns out after some algebra that the above q's satisfy a linear system of integro-differential equations, whose solution is most expediently expressed in terms of Laplace transforms, noted by carets (ˆ) on symbols, as follows:

$$\hat{q}_a(s) = -\frac{s^2}{s^2+\Omega^2}\sum_\alpha\left\{\sum_{b=1}^{N}\left[\delta_{ab}+\eta\,\frac{s^2}{s^2+\Omega^2}\sum_\nu\frac{\Omega^2}{s^2+\omega_\nu^2}\,\chi_{ab}^{(\nu)}\right]^{-1}\cdot\right.$$
$$\left.\cdot\left[\sum_\mu\frac{B_{\mu,b}^{(\alpha)}}{s^2+\omega_\mu^2}\right]\right\}\hat{g}^{(\alpha)}(s)\,, \qquad a=1,\ldots,N\,, \tag{4}$$

where μ,ν are multiple indices $\{n\ell m\}$ each, and $B^{(\alpha)}_{\mu,b}$ is a term proportional to overlapping integrals of the tidal form factors $\mathbf{f}^{(\alpha)}(\mathbf{x})$ —we skip too many technical details; $\chi^{(\nu)}_{ab}$ is a diadic product of sphere wave functions at locations a and b, and for a *perfect sphere* is given by

$$\chi^{(\nu)}_{ab} \equiv \chi^{(n\ell)}_{ab} = \frac{2\ell+1}{4\pi}\,A^2_{n\ell}(R)\,P_\ell(\mathbf{n}_a\!\cdot\!\mathbf{n}_b)\,, \tag{5}$$

with P_ℓ a Legendre polynomial and $A_{n\ell}(R)$ a *radial function* coefficient —see [1]. Equations (4) are somewhat complicated, but this is a result of their generality. We now examine their consequences in specific cases of interest.

In a practical GW antenna the resonators will be *much lighter* than the sphere, so the inequality $\eta \ll 1$ will safely hold. Also, their resonance frequency will be assumed to be *accurately tuned* to one of the sphere's eigenfrequencies, $\omega_{n\ell}$, say. What we do next is to expand $\hat{q}_a(s)$ as a *perturbative series* in η. Our attention is first drawn to recognise the presence of an *inverse matrix* in the rhs of (4): the *poles* of this inverse matrix, relative to the Laplace variable s, give the *characteristic frequencies* of the coupled system, while the corresponding *amplitudes* are given by the *residues* at those poles. The following is found: to lowest order in η, there appears a *symmetric multiplet* of frequencies around $\Omega = \omega_{n\ell}$, given by

$$\omega^2_{a\pm} = \omega^2_{n\ell}\left(1 \pm \sqrt{\frac{2\ell+1}{4\pi}}\;|A_{n\ell}(R)|\;\zeta_a\,\eta^{1/2}\right) + \mathrm{O}(\eta)\;, \qquad a = 1,\dots,N\,, \tag{6}$$

where ζ^2_a are the *eigenvalues* of the matrix $P_\ell(\mathbf{n}_a\!\cdot\!\mathbf{n}_b)$. Some of these may be multiple, and it is not difficult to prove that $0 < \zeta^2_a \le N$, except if $N > 2\ell+1$, in which case $2\ell+1$ of them still verify the same inequalities, while all the rest up to N are identically zero. Besides the multiplet (6), the system has *infinitely many* other characteristic frequencies, which are, respectively, equal to the sphere's frequencies non-equal to Ω, except that they are shifted relative to these by factors of order η rather than $\eta^{1/2}$, as in (6). The *amplitudes* associated to the latter are, however, seen to be smaller by factors of at least $\eta^{1/2}$ relative to the amplitudes of the modes in the multiplet (6); also, the amplitudes of the modes corresponding to the eventual eigenvalue $\zeta = 0$ in (6), which are shifted too by factors of order η relative to $\omega_{n\ell} = \Omega$, are seen to have the same kind of weaker coupling. These facts, which directly derive from the fundamental equations (4), constitute the *theoretical explanation* of the experimental observation that "the amount of frequency splitting is an indicator of the strength of the coupling" [3].

The *quantitative* formula for the amplitudes is

$$\hat{q}_a(s) = \eta^{-1/2}\sum_\alpha \Lambda^{(\alpha)}_a(s;n,\ell)\;\hat{g}^{(\alpha)}(s) + \mathrm{O}(\eta^0)\quad, \qquad a = 1,\dots,N\,, \tag{7}$$

where $\mathrm{O}(\eta^0)$ contains weaker coupling terms. The coefficient $\eta^{-1/2}$ accounts for the enhanced amplitude of the resonator motions due to resonant energy transfer from the much more massive sphere. $\Lambda^{(\alpha)}_a(s;n,\ell)$ is a *transfer function matrix*; it naturally depends on the mode (n,ℓ) chosen for tuning, and is also a *pattern matrix.* Equation (7) formally displays the fact that the number of resonators must be at least equal to the number of signal amplitudes $\hat{g}^{(\alpha)}(s)$ to be determined. In the case of a *general metric* GW this number is 6, as there is one monopole amplitude $\hat{g}^{(S)}(s)$ and five quadrupole amplitudes $\hat{g}^{(m)}(s)$ [1]. The matrix $\Lambda^{(\alpha)}_a(s;n,\ell)$ can be seen to be such that $\Lambda^{(S)}_a(s;n,\ell) \equiv 0$ unless $\ell = 0$, and $\Lambda^{(m)}_a(s;n,\ell) \equiv 0$ unless $\ell = 2$. In other words, if we want to measure the *scalar* amplitude $\hat{g}^{(S)}(s)$, then we need (at least) one transducer tuned to a sphere monopole frequency, ω_{n0}, and we cannot hope to measure it with a system tuned to other frequencies, *even if the signal $g^{(S)}(t)$ has significant*

Fourier components at those frequencies. Likewise, if we want to measure the quadrupolar $\hat{g}^{(m)}(s)$, then we *must* tune our (at least 5) resonators to a quadrupole sphere frequency, ω_{n2}.

The discussion of monopole radiation sensing is relatively straightforward, and independent of the transducer layout. The truly interesting issue is what happens to the quadrupole signal, so we concentrate on it for the rest of this paper. After rather laborious algebra, the Λ-matrix is found to be

$$\Lambda_a^{(m)}(s;n,2) \,=\, (-1)^N\,\sqrt{\frac{4\pi}{5}}\,b_n\,\sum_{b=1}^{N}\left\{\sum_{\zeta_c\neq 0}\frac{1}{2}\left[\left(s^2+\omega_{c+}^2\right)^{-1}-\left(s^2+\omega_{c-}^2\right)^{-1}\right]\frac{v_a^{(c)}v_b^{(c)*}}{\zeta_c}\right\}\cdot$$
$$\cdot\, Y_{2m}(\theta_b,\varphi_b)\,, \qquad (8)$$

where b_n is an overlapping integral ($b_1/R=0.328$, $b_2/R=0.106$ for the lowest modes —cf. [1]), and $v_a^{(c)}$ is the eigenvector corresponding to ζ_c^2. If Johnson and Merkowitz's TIGA is used in eq. (8), it then greatly simplifies to

$$\Lambda_a^{(m)}(s;n,2)=\sqrt{\frac{2\pi}{3}}\,b_n\,\frac{1}{2}\left[\left(s^2+\omega_+^2\right)^{-1}-\left(s^2+\omega_-^2\right)^{-1}\right]\,Y_{2m}(\theta_a,\varphi_a) \qquad (9)$$

as first seen by these authors [2]. The reason for this simplicity is that in such configuration there is *one* zero ζ-eigenvalue (since there are 6 resonators), whilst the other 5 are *all equal* ($\zeta_c^2=6/5$, $c=2,\ldots,6$), so the whole multiplet (6) collapses into a *single doublet*, $\omega_\pm$, plus the *weakly coupled* singlet at $\Omega=\omega_{n2}$. The nice properties of the rectangular matrix $Y_{2m}(\theta_a,\varphi_a)$ have also been exploited by Johnson and Merkovitz to define *mode channels*, which are direct measurements of the 5 wave amplitudes $\hat{g}^{(m)}(s)$. The "uniqueness" of the TIGA configuration derives from the fact that it is the *minimal one* (in number of resonators) having a *quintuple* ζ-eigenvalue, thus giving *isotropic sensitivity* to the sensor layout.

The generality of eq. (8) has allowed us for the first time to consistently explore other resonator layouts, some of which may be appealing alternatives to the TIGA. For example, any 5-resonator distribution with a *pentagonal symmetry axis* results in *three* frequency doublets, two of which are doubly degenerate. Actually, this is the system response for such category of configurations:

$$\hat{q}_a(s) \,=\, -\eta^{-1/2}\,\sqrt{\frac{4\pi}{5}}\,b_n\Bigg\{\,\frac{1}{2\zeta_0}\left[\left(s^2+\omega_{0+}^2\right)^{-1}-\left(s^2+\omega_{0-}^2\right)^{-1}\right]Y_{20}(\theta_a,\varphi_a)\,\hat{g}^{(0)}(s)\,+$$
$$+\frac{1}{2\zeta_1}\left[\left(s^2{+}\omega_{1+}^2\right)^{-1}{-}\left(s^2{+}\omega_{1-}^2\right)^{-1}\right]\left[Y_{21}(\theta_a,\varphi_a)\,\hat{g}^{(1)}(s){+}Y_{2-1}(\theta_a,\varphi_a)\,\hat{g}^{(-1)}(s)\right]+$$
$$+\frac{1}{2\zeta_2}\left[\left(s^2{+}\omega_{2+}^2\right)^{-1}{-}\left(s^2+\omega_{2-}^2\right)^{-1}\right]\left[Y_{22}(\theta_a,\varphi_a)\,\hat{g}^{(2)}(s){+}Y_{2-2}(\theta_a,\varphi_a)\,\hat{g}^{(-2)}(s)\right]\Bigg\}. \quad (10)$$

Although the structure of eq. (8) shows that there always exist "generalised mode channels", they are particularly simple in the case of *pentagonal* configurations. For these, the five mode channels are $\sum_{a=1}^5 Y_{2m}^*(\theta_a,\varphi_a)\,\hat{q}_a(s)$, $m=-2,\ldots,2$. The distinctive characteristic of these configurations is, as shown by eq. (10), that *different channels come at different frequencies*, *i.e.* the $\hat{g}^{(0)}(s)$ amplitude only excites the doublet $\omega_{0\pm}$, and is seen in mode channel 0; $\hat{g}^{(\pm1)}(s)$ can only excite the $\omega_{1\pm}$ doublet, and are seen in mode channels ±1, respectively; and, *mutatis mutandi*, the same for $\hat{g}^{(\pm2)}(s)$.

But this may be used to great advantage in a GW detector. For, imagine we *know* that a GW arrives down the resonators' symmetry axis. Now, evidence of excitation of the $\omega_{0\pm}$ or $\omega_{1\pm}$ frequency components, for example, is a *strong veto* on general relativity, as this theory predicts the excitation of *only* the ±2 modes. It is somehow unrealistic to think of having

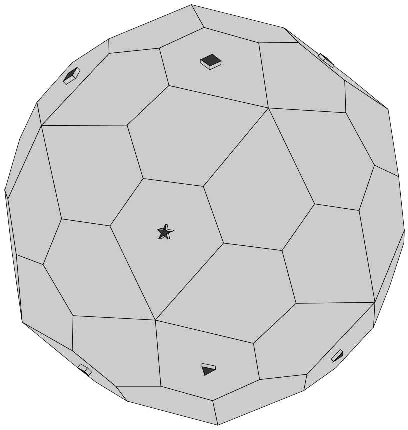

Fig. 1. – The proposed polyhedric antenna. Transducers are marked as follows: a *square* for the first quadrupole frequency, a *triangle* for the second, and a *star* for the monopole.

such information on incidence direction, but a more likely practical situation can also be handled advantageously. Indeed, the fact that different *wave modes* couple to different *detector frequencies* is a very powerful *discrimination* tool; at the same time, the frequency span of the multiplet in a forseeable GW antenna will only be a few tens of Hz, so the *signal spectrum* is likely to be constant over such span, and hence proposed deconvolution techniques [1], [5] comfortably applicable.

In fig. 1 we give a graphical representation of what might be considered an interesting *practical* implementation of a GW antenna based on the just-discussed pentagonal transducer layout. It relies on the philosophy of having a *polyhedron*, rather than a sphere, as a suitable approach to the GW spherical antenna, for ease of instrumentation attachment and manipulation [4]; the choice was made having in mind that the polyhedron should be as spherical as possible, whilst having at the same time axes of pentagonal symmetry. Our polyhedron is called *pentagonal hexacontahedron* [6], has sixty identical faces (irregular pentagons), and is considerably more spherical than the TI ([1]). An *inscribed* sphere exists which is tangent to every face at a point, to which a resonator could eventually be linked, thereby accomplishing a perfect simulation of a spherical distribution, *i.e.* all transducers *equidistant* from the centre.

In fig. 1 we also indicate proposed resonator locations —see caption for details. For example, for the *first* quadrupole resonance $\Omega = \omega_{02}$, it is found that ([2])

$$\omega_{0\pm} = \omega_{02}\left(1 \pm 0.5755\eta^{1/2}\right), \quad \omega_{1\pm} = \omega_{02}\left(1 \pm 0.8787\eta^{1/2}\right), \quad \omega_{2\pm} = \omega_{02}\left(1 \pm 1.0668\eta^{1/2}\right). \quad (11)$$

In addition to the set of five transducers tuned to ω_{02}, *another set* of five resonators, tuned to the *second* quadrupole frequency, ω_{12}, and located symmetrically in the "southern hemisphere", could be attached to the sphere, too. An *eleventh* resonator, tuned to the *first monopole* frequency, ω_{00}, and placed at an arbitrary position, could finally be added as well. Such an altogether 11-transducer configuration would take advantage of the large-sphere

([1]) For example, its volume is 1.057 times that of its inscribed sphere, while the truncated icosahedron is 1.153 times less voluminous than its circumscribed sphere; this means sphericity is a factor of almost 3 better for our polyhedron.

([2]) The chosen distribution has the property that the frequency spacing between members of the associated multiplet is the most *even* compatible with the polyhedron face orientations.

cross-section at its second quadrupole mode [7], and would, therefore, constitute a rather *complete* GW *detector* of its own. Also, it just requires 5 transducers rather than 6 for each quadrupole mode sensed.

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